Numerical solution of the two-dimensional nonlinear Schrodinger equation; homotopy perturbation method

B. Farokhi

Department of Physics, Faculty of Science, Arak University, 38156-8-8349, Arak, Iran.

Nonlinear partial differential equations have attracted a great deal of interest in recent years. Analytical solutions can also be obtained by different methods such as Adomian decomposition method, variational iteration method, homotopy analysis method and homotopy perturbation method (HPM). In the last method the solution is considered as a summation of an infinite series which usually converges rapidly to the exact solutions. The nonlinear Schrödinger equation (NLS) and the coupled nonlinear Schrödinger equation (CNLS) are one of the important partial differential equations which are often encountered in many branches of physics, chemistry and engineering, due to that, many researchers have motivated to solve them exactly or numerically.

Here, the numerical analysis of the coupled nonlinear Schrodinger equation (CNLS) is studied by the Homotopy Perturbation Method (HPM). The available analytical solution of one-dimensional CNLS obtained by Wadati *et. al.* is compared with HPM to examine the accuracy of the method.

Consider the non-linear differential equations, to illustrate the basic idea of the coupled HPM,

$$A(u,v) - f(r) = 0 \quad , \quad B(u,v) - g(r) = 0 \tag{1}$$

with the boundary condition $C(u,v,\partial u/\partial\hat{n},\partial v/\partial\hat{n})\big|_{on the boundary} = 0$, where A and B are general differential operators, C is the boundary operator. Assume that operators A and B have two; linear (L_1, L_2) and nonlinear (N_1, N_2)parts. One can rewrite the Eq. (1) as

$$L_1(u) + N_1(u,v) - f(r) = 0, \quad L_2(v) + N_2(u,v) - g(r) = 0 \tag{2}$$

Using the homotopy technique, we construct new functions $U(r,p)$, $V(r,p)$, which satisfies:

$$(1-p)[L_1(U) - L_1(u_0)] + p[A(U) - f(r)] = 0, \; (1-p)[L_2(V) - L_2(v_0)] + p[B(V) - g(r)] = 0 \tag{3}$$

where (u_0, v_0) is an initial approximation of (u,v) which satisfies the boundary conditions. From changing parameter p from zero to unity, the functions $U(r,p)$, $V(r,p)$ will change from u_0, v_0 to $u(r)$, $v(r)$. Now, assume that the solution can be written as a power series in p,

$$U = U_0 + pU_1 + p^2U_2 + \cdots, \quad V = V_0 + pV_1 + p^2V_2 + \cdots \tag{4}$$

Now, we will use this method for solving the following coupled nonlinear Schrodinger equations.

$$i\left(\Phi_t + a_1\Phi_x + a_2\Phi_y\right) + a_3\Phi_{xx} + a_4\Phi_{yy} + (a_5|\Phi|^2 + a_6|\Psi|^2)\Phi + (a_7|\Phi|^2 + a_8|\Psi|^2)\Psi = 0 \qquad \text{(5-a)}$$

$$i\left(\Psi_t + b_1\Psi_x + b_2\Psi_y\right) + b_3\Psi_{xx} + b_4\Psi_{yy} + \left(b_5|\Psi|^2 + b_6|\Phi|^2\right)\Phi + \left(b_7|\Psi|^2 + b_8|\Phi|^2\right)\Psi = 0 \qquad \text{(5-b)}$$

where $\Phi(x, y)$ and $\Psi(x, y)$ are the wave amplitudes in two dimensions and a_i, b_i are constant coefficients. By using $\Phi = u + iv$ and $\Psi = w + iz$ one can separate Eqs. (6) into the real and the imaginary parts. Therefore,

$$u_t + a_1u_x + a_2u_y + a_3v_{xx} + a_4v_{yy} + \left[a_5(u^2+v^2) + a_6(w^2+z^2)\right]v + \left[a_7(u^2+v^2) + a_8(w^2+z^2)\right]z = 0 \qquad (6)$$

$$-v_t - a_1v_x - a_2v_y + a_3u_{xx} + a_4u_{yy} + \left[a_5(u^2+v^2) + a_6(w^2+z^2)\right]u + \left[a_7(u^2+v^2) + a_8(w^2+z^2)\right]w = 0$$

$$w_t + b_1w_x + b_2w_y + b_3z_{xx} + b_4z_{yy} + \left[b_5(u^2+v^2) + b_6(w^2+z^2)\right]v + \left[b_7(u^2+v^2) + b_8(w^2+z^2)\right]z = 0$$

$$-z_t - b_1z_x - b_2z_y + b_3w_{xx} + b_4w_{yy} + \left[b_5(u^2+v^2) + b_6(w^2+z^2)\right]u + \left[b_7(u^2+v^2) + b_8(w^2+z^2)\right]w = 0$$

By defining new functions $U(r,p)$, $V(r,p)$, $W(r,p)$, $Z(r,p)$ which satisfies:

$$(1-p)(U-u_0)_t + p[U_t + a_1U_x + a_2U_y + a_3V_{xx} + a_4V_{yy} + a_5V(U^2+V^2) + a_6V(W^2+Z^2) + a_7Z(U^2+V^2) + a_8Z(W^2+Z^2)] = 0 \qquad (7)$$

$$-(1-p)(V+v_0)_t - p[V_t + a_1V_x + a_2V_y - a_3U_{xx} - a_4U_{yy} - a_5U(U^2+V^2) - a_6U(W^2+Z^2) - a_7W(U^2+V^2) - a_8W(W^2+Z^2)] = 0$$

$$(1-p)(W-w_0)_t + p[W_t + b_1W_x + b_2W_y + b_3Z_{xx} + b_4Z_{yy} + b_5V(U^2+V^2) + b_6V(W^2+Z^2) + b_7Z(U^2+V^2) + Zb_8(W^2+Z^2)] = 0$$

$$(1-p)(-Z+z_0)_t - p[Z_t + b_1Z_x + b_2Z_y - b_3W_{xx} - b_4W_{yy} + b_5U(U^2+V^2) + b_6U(W^2+Z^2) + b_7W(U^2+V^2) + b_8W(W^2+Z^2)] = 0$$

By substituting solutions (4), into Eqs. (7), and equating the coefficients of the terms with the identical powers of p,

$$p^0 \Rightarrow U_0 = u_0,\ V_0 = v_0,\ W_0 = w_0,\ Z_0 = z_0,$$

$$p^1 : \begin{cases} U_{1t} + a_1U_{0x} + a_2U_{0y} + a_3V_{0xx} + a_4V_{0yy} + \left[a_5(U_0^2+V_0^2) + a_6(W_0^2+Z_0^2)\right]V_0 \\ \quad + \left[a_7(U_0^2+V_0^2) + a_8(W_0^2+Z_0^2)\right]Z_0 = 0 \\ -V_{1t} - a_1V_{0x} - a_2V_{0y} + a_3U_{0xx} + a_4U_{0yy} + \left[a_5(U_0^2+V_0^2) + a_6(W_0^2+Z_0^2)\right]U_0 \\ + \left[a_7(U_0^2+V_0^2) + a_8(W_0^2+Z_0^2)\right]W_0 = 0 \\ W_{1t} + b_1W_{0x} + b_2W_{0y} + b_3Z_{0xx} + b_4Z_{0yy} + \left[b_5(U_0^2+V_0^2) + b_6(W_0^2+Z_0^2)\right]V_0 \\ \quad + \left[b_7(U_0^2+V_0^2) + b_8(W_0^2+Z_0^2)\right]Z_0 = 0 \\ -Z_{1t} - b_1Z_{0x} - b_2Z_{0y} + b_3W_{0xx} + b_4W_{0yy} + \left[b_5(U_0^2+V_0^2) + b_6(W_0^2+Z_0^2)\right]U_0 \\ + \left[b_7(U_0^2+V_0^2) + b_8(W_0^2+Z_0^2)\right]W_0 = 0 \end{cases}$$

$$p^2:\begin{cases} U_{2t}+a_1U_{1x}+a_2U_{1y}+a_3V_{1xx}+a_4V_{1yy}+\left[a_5(2U_0U_1+2V_0V_1)+a_6(2W_0W_1+2Z_0Z_1)\right]V_0 \\ +a_7[(2U_0U_1+2V_0V_1)+a_8(2W_0W_1+2Z_0Z_1)]Z_0+\left[a_5(U_0^2+V_0^2)+a_6(W_0^2+Z_0^2)\right]V_1 \\ +\left[a_7(U_0^2+V_0^2)+a_8(W_0^2+Z_0^2)\right]Z_1=0 \\ -V_{2t}-a_1V_{1x}-a_2V_{1y}+a_3U_{1xx}+a_4U_{1yy}+\left[a_5(2U_0U_1+2V_0V_1)+a_6(2W_0W_1+2Z_0Z_1)\right]U_0 \\ +[a_7(2U_0U_1+2V_0V_1)+a_8(2W_0W_1+2Z_0Z_1)]W_0+\left[a_5(U_0^2+V_0^2)+a_6(W_0^2+Z_0^2)\right]U_1 \\ +\left[a_7(U_0^2+V_0^2)+a_8(W_0^2+Z_0^2)\right]W_1=0 \\ W_{2t}+b_1W_{1x}+b_2W_{1y}+b_3Z_{1xx}+b_4Z_{1yy}+\left[b_5(2U_0U_1+2V_0V_1)+b_6(2W_0W_1+2Z_0Z_1)\right]V_0 \\ +b_7[(2U_0U_1+2V_0V_1)+b_8(2W_0W_1+2Z_0Z_1)]Z_0+\left[b_5(U_0^2+V_0^2)+b_6(W_0^2+Z_0^2)\right]V_1 \\ +\left[b_7(U_0^2+V_0^2)+b_8(W_0^2+Z_0^2)\right]Z_1=0 \\ -Z_{2t}-b_1Z_{1x}-b_2Z_{1y}+b_3W_{1xx}+b_4W_{1yy}+\left[b_5(2U_0U_1+2V_0V_1)+b_6(2W_0W_1+2Z_0Z_1)\right]U_0 \\ +[b_7(2U_0U_1+2V_0V_1)+b_8(2W_0W_1+2Z_0Z_1)]W_0+\left[b_5(U_0^2+V_0^2)+b_6(W_0^2+Z_0^2)\right]U_1 \\ +\left[b_7(U_0^2+V_0^2)+b_8(W_0^2+Z_0^2)\right]W_1=0 \end{cases}$$

where we will continue for finding other terms of serie.

For example, consider the propagation of pulses with equal mean frequencies in nonlinear fiber, which is governed by the coupled nonlinear Schrodinger equation [11],

$$i\left(\frac{\partial\Phi}{\partial t}+\eta\frac{\partial\Phi}{\partial x}\right)+\frac{1}{2}\frac{\partial^2\Phi}{\partial x^2}+\left(|\Phi|^2+e|\Psi|^2\right)\Phi=0, \tag{8-a}$$

$$i\left(\frac{\partial\Psi}{\partial t}-\eta\frac{\partial\Psi}{\partial x}\right)+\frac{1}{2}\frac{\partial^2\Psi}{\partial x^2}+\left(|\Psi|^2+e|\Phi|^2\right)\Psi=0, \tag{8-b}$$

where Φ, Ψ are the wave amplitudes in two polarizations and η is the normalized strength of the linear birefringence. Following the discussions of Wadati *et. al.* [12], the exact solution of Eq. (8) is

$$\Phi(x,t)=\sqrt{\frac{2\alpha}{1+e}}\,sech\left(\sqrt{2\alpha}(x-vt)\right)exp\left\{i\left[(v-\eta)x-0.5(v^2-\eta^2-2\alpha)t\right]\right\}, \tag{9-a}$$

$$\Psi(x,t)=\pm\sqrt{\frac{2\alpha}{1+e}}\,sech\left(\sqrt{2\alpha}(x-vt)\right)exp\left\{i\left[(v+\eta)x-0.5(v^2-\eta^2-2\alpha)t\right]\right\}, \tag{9-b}$$

where α and v depend on initial value (for simplicity $\alpha=v=1$). By assuming $\eta=0.5$ and $e=2/3$ in equation (8) and $a_1=-b_1=0.5$, $a_3=b_3=0.5$, $a_5=b_7=1$, $a_6=b_8=2/3$, $a_2=a_4=a_7=a_8=0$, $b_2=b_4=b_5=b_6=0$, in Eq. (5), both set of equations are the same. Solution of Eqs. (5) with initial conditions, $u_0=\sqrt{1.2}\,sech(\sqrt{2}x)cos(x/2)$ $v_0=\sqrt{1.2}\,sech(\sqrt{2}x)sin(x/2)$, $w_0=\sqrt{1.2}\,sech(\sqrt{2}x)cos(3/2x)$,

$z_0 = \sqrt{1.2}\, sech(\sqrt{2}x) sin(3/2x)$ and by homotopy technique is

$$\begin{cases} U_1 = t(-5/8\, sech(\sqrt{2}x) sin(x/2) + \sqrt{2}\, cos(x/2) sech(\sqrt{2}x) tanh(\sqrt{2}x)) \\ V_1 = t(5/8\, sech(\sqrt{2}x) cos(x/2) + \sqrt{2}\, sin(x/2) sech(\sqrt{2}x) tanh(\sqrt{2}x)) \\ W_1 = t(-5/8\, sech(\sqrt{2}x) sin(3x/2) + \sqrt{2}\, cos(3x/2) sech(\sqrt{2}x) tanh(\sqrt{2}x)) \\ Z_1 = t(5/8\, sech(\sqrt{2}x) cos(3x/2) + \sqrt{2}\, sin(3x/2) sech(\sqrt{2}x) tanh(\sqrt{2}x)) \end{cases}$$

$$\begin{cases} U_2 = \frac{1}{128} t^2 [-153\, cos(x/2) sech(\sqrt{2}x) - 80\sqrt{2}\, sech(\sqrt{2}x) sinh(x/2) tanh(\sqrt{2}x) + 256\, cos(x/2) sech(\sqrt{2}x) tanh(\sqrt{2}x)^2] \\ V_2 = \frac{1}{128} t^2 [-153\, sin(x/2) sech(\sqrt{2}x) + 80\sqrt{2}\, sech(\sqrt{2}x) cos(x/2) tanh(\sqrt{2}x) + 256\, sin(x/2) sech(\sqrt{2}x) tanh(\sqrt{2}x)^2] \\ W_2 = \frac{1}{128} t^2 [-153\, cos(3x/2) sech(\sqrt{2}x) - 80\sqrt{2}\, sech(\sqrt{2}x) sinh(3x/2) tanh(\sqrt{2}x) + 256\, cos(3x/2) sech(\sqrt{2}x) tanh(\sqrt{2}x)^2] \\ Z_2 = \frac{1}{128} t^2 [-153\, sin(3x/2) sech(\sqrt{2}x) + 80\sqrt{2}\, sech(\sqrt{2}x) cos(3x/2) tanh(\sqrt{2}x) + 256\, sin(3x/2) sech(\sqrt{2}x) tanh(\sqrt{2}x)^2] \end{cases}$$

$$\begin{cases} U_3 = t^3 [0.67\, sin(x/2) sech(\sqrt{2}x) - 1.86\sqrt{2}\, sech(\sqrt{2}x) cos(x/2) tanh(\sqrt{2}x) \\ \quad -1.25\, sin(x/2) sech(\sqrt{2}x) tanh(\sqrt{2}x)^2 + 2\sqrt{2}\, sech(\sqrt{2}x) cos(x/2) tanh(\sqrt{2}x)^3] \\ V_3 = t^3 [-0.67\, cos(x/2) sech(\sqrt{2}x) - 1.86\sqrt{2}\, sech(\sqrt{2}x) sin(x/2) tanh(\sqrt{2}x) \\ \quad -1.25\, cos(x/2) sech(\sqrt{2}x) tanh(\sqrt{2}x)^2 + 2\sqrt{2}\, sech(\sqrt{2}x) sin(x/2) tanh(\sqrt{2}x)^3] \\ W_3 = t^3 [0.67\, sin(3x/2) sech(\sqrt{2}x) - 1.86\sqrt{2}\, sech(\sqrt{2}x) cos(3x/2) tanh(\sqrt{2}x) \\ \quad -1.25\, sin(3x/2) sech(\sqrt{2}x) tanh(\sqrt{2}x)^2 + 2\sqrt{2}\, sech(\sqrt{2}x) cos(3x/2) tanh(\sqrt{2}x)^3] \\ Z_3 = t^3 \Big[2.64\, cos(3x+2y) sech[\sqrt{2}x - \sqrt{2}y] - 3.2\sqrt{2}\, sin(3x+2y) \\ \quad sech[\sqrt{2}x - \sqrt{2}y]\, tanh[\sqrt{2}x - \sqrt{2}y] - 3.5 sos(3x+2y) sech[\sqrt{2}x - \sqrt{2}y]\, tanh[\sqrt{2}x - \sqrt{2}y]^2 \\ \quad + 2\sqrt{2}\, sin(3x+2y) sech[\sqrt{2}x - \sqrt{2}y]\, tanh[\sqrt{2}x - \sqrt{2}y]^3 \Big] \end{cases}$$

The closed form of solution are the same as Eqs (10).

As a conclusion, we have developed the homotopy perturbation method (HPM). Numerical analysis of the coupled (one and two dimensions) nonlinear Schrodinger equation (CNLS) is studied by using the HPM. The available analytical solution of one-dimensional CNLS obtained by Wadati *et.al.* is compared with HPM to examine the accuracy of the method. The numerical results validate the convergence and accuracy of the HPM for analyzed CNLS.

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