Simulation of two-dimensional transport in tokamak plasmas for integrated analysis of core and peripheral plasmas

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Introduction

In most conventional transport simulations of tokamak plasmas, transport in the core region is usually described as a one-dimensional problem by flux-surface average, since transport along a field line is very fast. On the other hand, transport in the peripheral SOL-divertor region is usually described as a two-dimensional problem with simplified transport models and plasma flow, since variation along the field line is considerable. For integrated modeling of both core and peripheral plasmas, however, a two-dimensional description of transport phenomena over the entire tokamak is desirable and is becoming feasible owing to recent progress in computational performance. Such two-dimensional transport modeling will make a more accurate evaluation of the confinement property including the pedestal region possible. In this study, we formulate a set of two-dimensional transport equations including the neoclassical viscous force [1] in the magnetic flux coordinates necessary for developing a two-dimensional transport simulation code TASK/T2.

Coordinate systems and assumptions

We employ magnetic flux coodinate system (MFCS), $(\xi_1^{\mathrm{M}}, \xi_2^{\mathrm{M}}, \xi_3^{\mathrm{M}}) = (\rho, \chi, \zeta)$ and $v_i^{\mathrm{M}} \equiv \vec{v} \cdot \nabla \xi_i^{\mathrm{M}}$, to express spatial variations of quantities in MHD equilibrium and neoclassical tranport coordinate system (NTCS), $(\vec{e}_1^{\mathrm{N}}, \vec{e}_2^{\mathrm{N}}, \vec{e}_3^{\mathrm{N}}) = (\nabla\rho, \vec{B}/B, R^2\nabla\zeta)$ and $v_i^{\mathrm{N}} \equiv \vec{v} \cdot \vec{e}_i^{\mathrm{N}}$, to express components of vector quantities for the compatibility with the neoclassical transport theory [1] as shown in Figure 1.

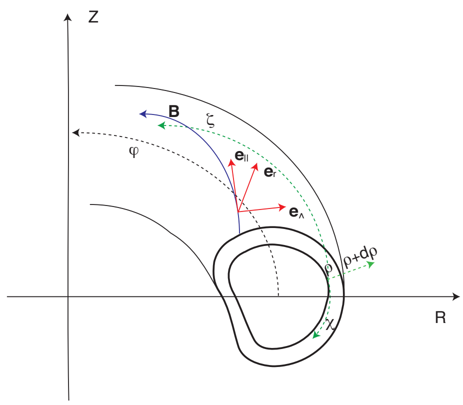

Figure 1: Coodinate systems

In this study the following five assumptions are made: 1) toroidal axisymmetry, 2) quantities related to MHD equilibrium depend only on the flux label ρ, 3) phenomena at the Alfvén velocity are much faster than diffusion of the magnetic field and transport phenomena, 4) force balance in the radial direction in the transport time scale, 5) weak time dependence of the basis vectors.

Derivation of the transport equations

We derive the two-dimensional transport model from fluid equations which consist of equation of continuity, equation of motion and equation of energy transport.

$$\frac{\partial n_a}{\partial t} + \nabla \cdot (n_a \vec{u}_a) = S_{na} \tag{1}$$

$$\frac{\partial}{\partial t}(m_a n_a \vec{u}_a) + \nabla \cdot (m_a n_a \vec{u}_a \vec{u}_a) = -\nabla p_a - \nabla \cdot \overleftrightarrow{\pi}_a + e_a n_a \left(\vec{E} + \vec{u}_a \times \vec{B} \right) + \vec{F}_a + \vec{S}_{ma} \tag{2}$$

$$\frac{\partial}{\partial t}\left(\frac{3}{2} p_a\right) + \nabla \cdot \left(\vec{q}_a + \frac{5}{2} p_a \vec{u}_a + \overleftrightarrow{\pi}_a \cdot \vec{u}_a \right) = \left(\nabla \cdot \overleftrightarrow{\pi}_a \right) \cdot \vec{u}_a + \vec{u}_a \cdot \nabla p_a + Q_a + S_{Ea} - \frac{1}{2} m_a u_a^2 S_{na} \tag{3}$$

where a denotes particle species, n_a is the density of particle, $\vec{u}_a$ is the flow velocity, S_{n_a} is the source of particle, m_a is the mass, p_a is the isotropic pressure, $\overleftrightarrow{\pi}_a$ is the anisotropic pressure tensor, e_a is the electric charge, $\vec{E}$ is the electric field, $\vec{B}$ is the magnetic field, $\vec{F}_a$ is the friction force, $\vec{S}_{ma}$ is the source of momentum, $\vec{q}_a$ is the heat flux, Q_a is the energy exchange term, and S_{Ea} is the source of energy. In this study, we employ the neoclassical viscosity tensor [1] as the the anisotropic pressure tensor

$$\overleftrightarrow{\pi}_a = \pi_a^{\text{neo}} \left(\vec{e}_2^{\,\text{N}} \vec{e}_2^{\,\text{N}} - \frac{1}{3} \overleftrightarrow{I} \right), \quad \pi_a^{\text{neo}} = p_{a\parallel} - p_{a\perp} \tag{4}$$

where π_a^{neo} is the neoclassial parallel viscosity coefficient.

The transport model consists of the equation of particle density, momentum and internal energy for electrons and ions. The equation for particle density is expressed as

$$\frac{\partial n_a}{\partial t} + \sum_{i=1}^{2} \sum_{j=1}^{3} \frac{1}{\sqrt{g}} \frac{\partial}{\partial \xi_i^{\text{M}}} \left(\sqrt{g} \mathscr{T}_{ij}^{\text{MN}} n_a u_{aj}^{\text{N}} \right) = S_{n_a} \tag{5}$$

where $\sqrt{g}$ is the jacobian of MFCS and $\mathscr{T}_{ij}^{\text{MN}}$ is the transform matrix from NTCS to MFCS. Taking the scalar product of Eq.(2) with $\vec{e}_1^{\,\text{N}}$ with the fourth assumption, we obtain the force balance equation in the radial direction

$$\begin{aligned} 0 = -\sum_{i=1}^{2} g^{1i} \frac{\partial p_a}{\partial \xi_i^{\text{M}}} + \sum_{i=1}^{2} \frac{1}{3} g^{1i} \frac{\partial \pi_a^{\text{neo}}}{\partial \xi_i^{\text{M}}} - \pi_a^{\text{neo}} \kappa_1^{\text{N}} - \sum_{i=1}^{2} g^{1i} e_a n_a \frac{\partial \phi}{\partial \xi_i^{\text{M}}} \\ + \frac{e_a B I}{\psi'} n_a u_{a2}^{\text{N}} - \frac{e_a B^2}{\psi'} n_a u_a^{\xi_3^{\text{N}}} + F_{a1}^{\text{N}} + S_{ma1}^{\text{N}} \end{aligned} \tag{6}$$

where g^{ij} is the contravariant metric tensor of MFCS, $\vec{\kappa}$ is the curvature vector of the magnetic field, ϕ is the electrostatic potential, I is the toroidal current, and ψ is the poloidal flux, ($\psi' \equiv d\psi/d\xi_1^{\text{M}}$). Taking the scalar products of Eq.(2) with $\vec{e}_2^{\,\text{N}}$, we obtain the equation for momentum in the parallel direction

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(m_a n_a u^{\mathrm{N}}_{a2}\right) - \frac{\sqrt{g}\psi' g^{12}}{BR^2}\sum_{i=1}^{2}\sum_{j=1}^{3}\frac{1}{\sqrt{g}}\frac{\partial}{\partial \xi^{\mathrm{M}}_i}\left(\sqrt{g}\mathscr{T}^{\mathrm{MN}}_{ij} m_a n_a u^{\mathrm{N}}_{a1} u^{\mathrm{N}}_{aj}\right)\\
&\qquad + \frac{\sqrt{g}\psi' g^{11}}{BR^2}\sum_{i=1}^{2}\sum_{j=1}^{3}\sum_{k=1}^{3}\frac{1}{\sqrt{g}}\frac{\partial}{\partial \xi^{\mathrm{M}}_i}\left(\sqrt{g}\mathscr{T}^{\mathrm{MN}}_{ij}\mathscr{T}^{\mathrm{MN}}_{2k} m_a n_a u^{\mathrm{N}}_{aj} u^{\mathrm{N}}_{ak}\right)\\
&\qquad + \frac{I}{BR^2}\sum_{i=1}^{2}\sum_{j=1}^{3}\frac{1}{\sqrt{g}}\frac{\partial}{\partial \xi^{\mathrm{M}}_i}\left(\sqrt{g}\mathscr{T}^{\mathrm{MN}}_{ij} m_a n_a u^{\mathrm{N}}_{a3} u^{\mathrm{N}}_{aj}\right) + \frac{\psi'}{\sqrt{g}B}\sum_{i=1}^{3}\sum_{j=1}^{3} C^{\mathrm{momentum}}_{ij} m_a n_a u^{\mathrm{N}}_{ai} u^{\mathrm{N}}_{aj}\\
&\qquad = -\frac{\psi'}{\sqrt{g}B}\frac{\partial p_a}{\partial \xi^{\mathrm{M}}_2} + \frac{\psi'}{\sqrt{g}B}\pi^{\mathrm{neo}}_a\frac{\partial \ln B}{\partial \xi^{\mathrm{M}}_2} - \frac{2}{3}\frac{\psi'}{\sqrt{g}B}\frac{\partial \pi^{\mathrm{neo}}_a}{\partial \xi^{\mathrm{M}}_2} - \frac{e_a n_a \psi'}{\sqrt{g}B}\frac{\partial \phi}{\partial \xi^{\mathrm{M}}_2} + F^{\mathrm{N}}_{a2} + S^{\mathrm{N}}_{ma2}
\end{aligned} \tag{7}$$

where $C^{\mathrm{momentum}}_{ij}$ is the geometric coefficient. In the derivation of Eq.(7) we used the following relation [2] for an arbitrary symmetric tensor $\overleftrightarrow{T}_S$.

$$\vec{e}^{\,\mathrm{N}}_3\cdot\left(\nabla\cdot\overleftrightarrow{T}_S\right) = \nabla\cdot\left(\vec{e}^{\,\mathrm{N}}_3\cdot\overleftrightarrow{T}_S\right) \tag{8}$$

Taking the scalar product of Eq.(2) with $\vec{e}^{\,\mathrm{N}}_3$, we obtain the equation for momentum in the toroidal direction

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(m_a n_a u^{\mathrm{N}}_{a3}\right) + \sum_{i=1}^{2}\sum_{j=1}^{3}\frac{1}{\sqrt{g}}\frac{\partial}{\partial \xi^{\mathrm{M}}_i}\left(\sqrt{g}\mathscr{T}^{\mathrm{MN}}_{ij} m_a n_a u^{\mathrm{N}}_{a3} u^{\mathrm{N}}_{aj}\right)\\
&\qquad = -\frac{1}{\sqrt{g}}\frac{\partial}{\partial \xi^{\mathrm{M}}_2}\left(\frac{I\psi'}{B^2}\pi^{\mathrm{neo}}_a\right) + e_a\psi' n_a u^{\mathrm{N}}_{a1} + F^{\mathrm{N}}_{a3} + S^{\mathrm{N}}_{ma3}
\end{aligned} \tag{9}$$

In the derivation of the equation for internal energy, we use following relation [3] derived from the conservation of energy through collisions among different particle species.

$$\vec{u}_{\mathrm{e}}\cdot\nabla p_{\mathrm{e}} + \left(\nabla\cdot\overleftrightarrow{\pi}_{\mathrm{e}}\right)\cdot\vec{u}_{\mathrm{e}} + Q_{\mathrm{e}} = -\sum_{a\neq \mathrm{e}}\left[\vec{u}_a\cdot\nabla p_a + \left(\nabla\cdot\overleftrightarrow{\pi}_a\right)\cdot\vec{u}_a + Q_a\right] + \vec{j}\cdot\vec{E} \tag{10}$$

Substituting this relation into Eq.(3), we obtain equation for internal energy for electron and ions as follows

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(\frac{3}{2}p_{\mathrm{e}}\right) - \sum_{i=1}^{2}\sum_{j=1}^{2}\frac{1}{\sqrt{g}}\frac{\partial}{\partial \xi^{\mathrm{M}}_i}\left(\sqrt{g}n_{\mathrm{e}}\chi^{\mathrm{M}}_{\mathrm{e}ij}\frac{\partial T_{\mathrm{e}}}{\partial \xi^{\mathrm{M}}_j}\right) + \sum_{i=1}^{2}\sum_{j=1}^{3}\frac{1}{\sqrt{g}}\frac{\partial}{\partial \xi^{\mathrm{M}}_i}\left(\sqrt{g}\frac{5}{2}\mathscr{T}^{\mathrm{MN}}_{ij}u^{\mathrm{N}}_{\mathrm{e}j}p_{\mathrm{e}}\right)\\
&\quad - \sum_{i=1}^{2}\sum_{j=1}^{3}\frac{1}{\sqrt{g}}\frac{\partial}{\partial \xi^{\mathrm{M}}_i}\left(\sqrt{g}\frac{1}{3}\pi^{\mathrm{neo}}_{\mathrm{e}}\mathscr{T}^{\mathrm{MN}}_{ij}u^{\mathrm{N}}_{\mathrm{e}j}\right) + \frac{1}{\sqrt{g}}\frac{\partial}{\partial \xi^{\mathrm{M}}_2}\left(\sqrt{g}\frac{\pi^{\mathrm{neo}}_{\mathrm{e}}\psi'}{\sqrt{g}B}u^{\mathrm{N}}_{\mathrm{e}2}\right)\\
&= -\sum_{a=\mathrm{ions}}\Bigg[-\sum_{i=1}^{2}\sum_{j=1}^{3}\frac{1}{3}\mathscr{T}^{\mathrm{MN}}_{ij}u^{\mathrm{N}}_{aj}\frac{\partial \pi^{\mathrm{neo}}_a}{\partial \xi^{\mathrm{M}}_i} - \left(\frac{\pi^{\mathrm{neo}}_a\psi'}{\sqrt{g}B}\frac{\partial \ln B}{\partial \xi^{\mathrm{M}}_2} - \frac{\psi'}{\sqrt{g}B}\frac{\partial \pi^{\mathrm{neo}}_a}{\partial \xi^{\mathrm{M}}_2}\right)u^{\mathrm{N}}_{a2}\\
&\qquad\qquad + \sum_{i=1}^{3}\sum_{j=1}^{3}\pi^{\mathrm{neo}}_a C^{\mathrm{energy}}_{ij}u^{\mathrm{N}}_{ai}\kappa^{\mathrm{M}}_j + \sum_{i=1}^{2}\sum_{j=1}^{3}\mathscr{T}^{\mathrm{MN}}_{ij}u^{\mathrm{N}}_{aj}\frac{\partial p_a}{\partial \xi^{\mathrm{M}}_i} + \frac{3}{2}n_a\frac{T_{\mathrm{e}} - T_a}{\tau_{a\mathrm{e}}}\Bigg]\\
&\quad - \sum_{a=\mathrm{e,i}}\sum_{i=1}^{2}\sum_{j=1}^{3} e_a\mathscr{T}^{\mathrm{MN}}_{ij}n_a u^{\mathrm{N}}_{aj}\frac{\partial \phi}{\partial \xi^{\mathrm{M}}_i} + S_{E\mathrm{e}} - \frac{1}{2}m_{\mathrm{e}}u^2_{\mathrm{e}}S_{n\mathrm{e}} \qquad \text{for electron}
\end{aligned} \tag{11}$$

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left(\frac{3}{2}p_{\mathrm{i}}\right)-\sum_{i=1}^{2}\sum_{j=1}^{2}\frac{1}{\sqrt{g}}\frac{\partial}{\partial\xi_i^{\mathrm{M}}}\left(\sqrt{g}n_{\mathrm{i}}\chi_{\mathrm{i}ij}^{\mathrm{M}}\frac{\partial T_{\mathrm{i}}}{\partial\xi_j^{\mathrm{M}}}\right)+\sum_{i=1}^{2}\sum_{j=1}^{3}\frac{1}{\sqrt{g}}\frac{\partial}{\partial\xi_i^{\mathrm{M}}}\left(\sqrt{g}\frac{5}{2}\mathscr{T}_{ij}^{\mathrm{MN}}u_{\mathrm{i}j}^{\mathrm{N}}p_{\mathrm{i}}\right)\\
&\quad-\sum_{i=1}^{2}\sum_{j=1}^{3}\frac{1}{\sqrt{g}}\frac{\partial}{\partial\xi_i^{\mathrm{M}}}\left(\sqrt{g}\frac{1}{3}\pi_{\mathrm{i}}^{\mathrm{neo}}\mathscr{T}_{ij}^{\mathrm{MN}}u_{\mathrm{i}j}^{\mathrm{N}}\right)+\frac{1}{\sqrt{g}}\frac{\partial}{\partial\xi_2^{\mathrm{M}}}\left(\sqrt{g}\frac{\pi_{\mathrm{i}}^{\mathrm{neo}}\psi'}{\sqrt{g}B}u_{\mathrm{i}2}^{\mathrm{N}}\right)\\
&=-\sum_{i=1}^{2}\sum_{j=1}^{3}\frac{1}{3}\mathscr{T}_{ij}^{\mathrm{MN}}u_{\mathrm{i}j}^{\mathrm{N}}\frac{\partial\pi_{\mathrm{i}}^{\mathrm{neo}}}{\partial\xi_i^{\mathrm{M}}}-\left(\frac{\pi_{\mathrm{i}}^{\mathrm{neo}}\psi'}{\sqrt{g}B}\frac{\partial\ln B}{\partial\xi_2^{\mathrm{M}}}-\frac{\psi'}{\sqrt{g}B}\frac{\partial\pi_{\mathrm{i}}^{\mathrm{neo}}}{\partial\xi_2^{\mathrm{M}}}\right)u_{\mathrm{i}2}^{\mathrm{N}}+\sum_{i=1}^{3}\sum_{j=1}^{3}\pi_{\mathrm{i}}^{\mathrm{neo}}C_{ij}^{\mathrm{energy}}u_{\mathrm{i}i}^{\mathrm{N}}\kappa_j^{\mathrm{M}}\\
&\quad+\sum_{i=1}^{2}\sum_{j=1}^{3}\mathscr{T}_{ij}^{\mathrm{MN}}u_{\mathrm{i}j}^{\mathrm{N}}\frac{\partial p_{\mathrm{i}}}{\partial\xi_i^{\mathrm{M}}}+\sum_{a\neq\mathrm{i}}\frac{3}{2}n_{\mathrm{i}}\frac{T_a-T_{\mathrm{i}}}{\tau_{\mathrm{i}a}}+S_{E\mathrm{i}}-\frac{1}{2}m_{\mathrm{i}}u_{\mathrm{i}}^2S_{n\mathrm{i}}\qquad\text{for ions}\qquad(12)
\end{aligned}
$$

In the derivation of Eq.(11) and (12), we employed Braginskii's heat flux $\vec{q}_a$ [4] in MFCS and the heat exchange term Q_a as follows.

$$
\vec{q}_a=-n_a\chi_{a\parallel}\nabla_{\parallel}T_a-n_a\chi_{a\wedge}\vec{e}_{\parallel}\times\nabla T_a-n_a\chi_{a\perp}\nabla_{\perp}T_a=-n_a\overleftrightarrow{\chi}_a\cdot\nabla T_a \qquad (13)
$$

$$
Q_a=\sum_{b\neq a}Q_{ab},\quad Q_{ab}\equiv\frac{3}{2}n_a\frac{T_b-T_a}{\tau_{ab}} \qquad (14)
$$

To close our transport model, appropriate two-dimensional modelings on the friction force $\vec{F}_a$ and the neoclassical viscosity coefficient π_a^{neo} are needed. We employ Braginskii's friction force [4] and neoclassical viscosity coefficient in high collisionality region in this formulation.

Conclusion

A set of equations required for two-dimensional transport modeling for tokamak plasmas has been derived for integrated analysis of core and peripheral plasmas. Transport equations have been derived with the neoclassical viscosity in MFCS and reduced to two-dimensional by toroidal axisymmetry. By combining these transport equations with the set of electromagnetic equations which consist of the equation for static electric field, the magnetic diffusion equation and Grad-Shafranov equation, a self-consistent two-dimensional transport analysis including the field evolution and a more accurate evaluation of the confinement property will be available.

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