Shafranov shift at strong plasma anisotropy in tokamaks and stellarators

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1. Introduction. The results obtained in tokamaks JET, Tore Supra, spherical tokamak MAST and in stellarators CHS and LHD show that, with intensive heating of the plasma, strong anisotropy of the pressure can occur [1–5]. In this case, the standard theory of plasma equilibrium with isotropic pressure p [6, 7] is not applicable. It is generally accepted that with $p_\parallel \neq p_\perp$ (pressure along and perpendicular to the magnetic field $\mathbf{B}$) the predictions of this theory remain valid, if the replacement $p \to 0.5(p_\parallel + p_\perp)$ is made in the expressions for the Shafranov shift Δ and the Pfirsch-Schlüter current [2–4, 6, 8]. In [6, 8] the equation for Δ was derived assuming weak poloidal dependence of $p_\parallel$ and $p_\perp$. Here, on the contrary, we assume strong poloidal modulation of $p_\perp$, which may be possible with non-central heating.

2. Formulation of the problem. The plasma equilibrium is described by the equation

$$0 = -\nabla \cdot \vec{\vec{p}} + \mathbf{j} \times \mathbf{B}, \tag{1}$$

where $\mathbf{j} = \nabla \times \mathbf{B}$ is the current density and $\vec{\vec{p}}$ is the pressure tensor,

$$\vec{\vec{p}} = p_\parallel \frac{\mathbf{BB}}{\mathbf{B}^2} + p_\perp \left(\vec{\vec{\mathbf{I}}} - \frac{\mathbf{BB}}{\mathbf{B}^2} \right), \tag{2}$$

where $\vec{\vec{\mathbf{I}}}$ is the unit dyadic. Following [6–8], equations for Δ in tokamaks and stellarators are derived using the model of shifted circular magnetic surfaces:

$$r = R_0 + \Delta(a) - a\cos\theta, \qquad z = a\sin\theta. \tag{3}$$

Here (r, ζ, z) are the cylindrical coordinates, R_0 is the distance from the axis of the system to the center of the vacuum chamber, a is the small radius of the magnetic surface, and θ is the poloidal angle.

3. Parameterization of the pressure. The parallel component of (2) gives us

$$\mathbf{B} \cdot \nabla (p_\parallel / B) = p_\perp \mathbf{B} \cdot \nabla (1/B). \tag{4}$$

We assume, as it is often done in theory, $p_\parallel = p_\parallel(a, B)$, $p_\perp = p_\perp(a, B)$. With

$$p_\perp = p_{\perp 0}(a) + p_{\perp 1}(a)(B_m / B - 1), \tag{5}$$

where $B_m = B_m(a)$ on the surface $a = const$, from (4) we obtain

$$p_{\|} = p_{\|0}(a) + (p_{\|0} - p_{\perp 0})(B_m/B - 1) + 0.5 p_{\perp 1}(B - B_m)^2/BB_m \,. \tag{6}$$

These are the exact solutions of equation (4). They are valid for any geometry, not only for the case (3). For circular magnetic surfaces (3) and $\mathbf{B}_m^2 \equiv \max \mathbf{B}^2$ these functions give

$$p_{\perp} \approx p_{\perp 0}(a) + p_{\perp as}(a)\sin^2\frac{\theta}{2}, \tag{7}$$

$$p_{\|} \approx p_{\|0}(a) - (p_{\|0} - p_{\perp 0})(2a/R_0)\sin^2\frac{\theta}{2} + p_{\perp as}(2a/R_0)\sin^4\frac{\theta}{2}, \tag{8}$$

where $p_{\perp as} = 2ap_{\perp 1}/R_0$. Note that $p_{\perp as} \ge -p_{\perp 0}$ since $p_{\perp} \ge 0$. Expressions (7) and (8) allow description of the profiles with a maximum at low ($p_{\perp as} > 0$) and high ($p_{\perp as} < 0$) field sides.

In contrast to the analysis in [6, 8], we assume that the poloidal variation of $p_{\perp}$, which is described by the term with $p_{\perp as}$ in (7), can be comparable to $p_{\perp 0}$.

4. Two-dimensional scalar equilibrium equation. In an axially symmetric system, the magnetic field can be written as ($2\pi\psi$ is the poloidal flux)

$$\mathbf{B} = \nabla\psi \times \nabla\zeta + F\nabla\zeta \,. \tag{9}$$

From the perpendicular component of (1) it follows that $\sigma F = F_K(\psi)$. Also, from (1) one can obtain the well-known (see, for example, [6, 8]) scalar two-dimensional equilibrium equation

$$\operatorname{div}\frac{\sigma\nabla\psi}{r^2} = -\frac{\partial p_{\|}(\psi, B)}{\partial\psi} - \frac{F_K F_K'}{\sigma r^2}, \tag{10}$$

where $\sigma = 1 - \sigma_{\|}$, $\sigma_{\|} = (p_{\|} - p_{\perp})/\mathbf{B}^2$. A similar equation can be obtained for stellarators, if ψ is replaced by $\psi - \psi_v$ on the left-hand side of (10) [8], where $2\pi\psi_v$ is the average vacuum poloidal flux due to the helical field [7, 8]. In the simplest case, the additional terms on the right-hand side [8] can be neglected.

Equation for Δ has been derived earlier from (1)–(3) for the plasma with $|p_{\perp as}| << p_{\perp 0}$ [6, 8]. Here, in contrast, $p_{\perp as}$ is not considered small. We use the consequence of (10):

$$\sigma r^2 \operatorname{div}\frac{\sigma\nabla\psi}{r^2} - \left\langle \sigma r^2 \operatorname{div}\frac{\sigma\nabla\psi}{r^2} \right\rangle = \left\langle \sigma r^2 \frac{\partial p_{\|}(\psi, B)}{\partial\psi} \right\rangle - \sigma r^2 \frac{\partial p_{\|}(\psi, B)}{\partial\psi}, \tag{11}$$

where the brackets $\langle ... \rangle$ denote the averaging over the volume V between two neighboring magnetic surfaces:

$$\langle f \rangle = \frac{d}{dV}\int f\sqrt{g}\,da\,d\theta\,d\zeta = \frac{2\pi}{V'}\int_0^{2\pi} afr(1 - \Delta'\cos\theta)d\theta\,, \tag{12}$$

where $\sqrt{g}$ is the Jacobian of the transformation of Cartesian to the flux coordinates (a,θ,z), the prime means the radial derivative. The second equality with $V' = 4\pi^2 a(R_c + a\Delta'/2)$ in (12) corresponds to the model of circular magnetic surfaces (3), which we use further. In the calculations we apply the standard equilibrium theory expansions in the small parameters a/R_0, $|\Delta'|$, $|a\Delta''|$, B_p/B_0 (the ratio of the poloidal field to toroidal one) keeping only linear terms in the expansions. This corresponds to the generally accepted approaches in the equilibrium theory [6, 7].

5. Equation for the Shafranov shift. With $p_\parallel$ given by (6), from (11) we obtain

$$[(\mu_J^2 a^3 \Delta')'/a^2 + \mu_J^2 a/R_0] B_0^2/R_0 = p'_{\parallel 0} + p'_{\perp 00}\,, \tag{13}$$

where μ_J is the current rotational transformation, B_0 is the toroidal field, and

$$p_{\perp 00} \equiv \frac{1}{2\pi}\int_0^{2\pi} p_\perp d\theta = p_{\perp 0} + 0.5 p_{\perp as}\,. \tag{14}$$

The second equality in (14) is valid for $p_\perp$ in the form (7).

For stellarators, we use the equation from [8] similar to (10). In the equation for Δ obtained from it, a part containing the pressure will be the same as the right-hand side of equation (13). Calculation of the left-hand side gives us the same result as in [7, 8]. Finally, we obtain a generalization of (13):

$$\frac{B_0^2}{R_0}(\mu_h + \mu_J)\left\{\frac{(\mu_J^2 a^3 \Delta')'}{\mu_J a^2} + \frac{a}{R_0}\mu_J + \frac{[(\mu_h \Delta)' a^3]'}{a^2}\right\} = p'_{\parallel 0} + p'_{\perp 00}\,, \tag{15}$$

here μ_h is the vacuum rotational transform produced by the helical fields.

Equations (13) and (15) coincide with the known equations for Δ [6, 8]. Thereby we have shown that equations (13) and (15) can be used beyond the restrictions imposed in [6, 8] where the poloidal dependence of the pressure was considered small. In other words, equations (13) and (15) are applicable even if there is a strong poloidal modulation of $p_\perp$, for example, due to non-central intensive heating [10, 11].

6. Discussion. The effect of the pressure anisotropy on the MHD equilibrium, especially on the position of the magnetic axis, has attracted attention in connection with experiments on the Large Helical Device (LHD) [4]. Our analytical treatment confirms the results of numerical calculations [4]. In particular, we have shown that Δ is determined by $(p'_{\parallel 0} + p'_{\perp 00})$ even in the presence of a significant modulation of $p_\perp$. This explains stronger correlation of the magnetic axis shift with $\beta_{eq} \sim (p_\parallel + p_\perp)$ than with $\beta \sim (p_\parallel + 2p_\perp)$ found in numerical calculations [4, 10]. The demonstrated applicability of equation (13) [6] and (15) [8] for a plasma with strong poloidal modulation of $p_\perp$ can also explain the result of [4] that the pressure difference from its average on the magnetic surface does not affect the Shafranov shift (Figs. 3–5 in [4]). It is known [9, 10] that, in conventional tokamaks and stellarators, the poloidal variation of $p_\parallel$ must be small irrespective of modulation of $p_\perp$. Therefore, at small $p_\perp$ or $p_\parallel > p_\perp$ the variation in pressure on the magnetic surface should not significantly affect the displacement of the magnetic axis, even if the next harmonics with $\cos 2\theta$, $\cos 3\theta$ etc. are taken into account as in [12], which also explains the results of [4].

Our conclusion about the applicability of equation (13) [6] in case of strong poloidal modulation of $p_\perp$ is also consistent with calculations in [11], where the simulations of the anisotropic plasma equilibrium in the JET tokamak with $p_\perp^{\max} / p_\perp^{\min} > 6$ on a magnetic surface (Fig 1 in [11]) have shown that Δ does not depend on $p_\perp / p_\parallel$ at fixed $\alpha \sim p'_{\parallel 0} + p'_{\perp 00}$. Therefore, it can be difficult to detect the poloidal asymmetry of $p_\perp$ by magnetic diagnostics.

There is a great difference between the equilibria with $p_\parallel >> p_\perp$ and $p_\parallel << p_\perp$. In the latter case, the most interesting and different from the studied earlier weak-anisotropy case [6, 8] is the equilibrium with strong poloidal modulation of $p_\perp$. This is practically unexplored situation, with only known analytical approach to the problem in [12]. Here we applied a different technique and used less restrictive assumptions than in [12].

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