Multi-fluid transport equations on the flux coordinates in tokamaks

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Abstract

The governing equations for a multi-fluid transport code TASK/TX, essentially based on a self-consistent two-fluid model, are derived on the axisymmetric flux coordinates.

Introduction

We have developed a one-dimensional fluid-type transport code TASK/TX [1]. The code is essentially based on a self-consistent two-fluid model (cf. [2]), which consists of two-fluid equations (conservation of mass, momentum and energy) plus Maxwell's equations. It also involves the equations for beam ions [3, 4] and neutrals [5]. It differs from conventional diffusive transport codes mainly in that: (1) it does not require the explicit quasi-neutrality condition $n_\mathrm{e} = \sum_i Z_i n_i$, but solves the momentum equations for electrons as well with Gauss's law (Poisson's equation); (2) the neoclassical quantities such as a particle flux, the bootstrap current and the resistivity are consistently calculated in an implicit manner that neoclassical viscosities calculated by NCLASS [6] and classical friction forces are embedded in the momentum equations [7]; and (3) a particle flux is not described as an explicit convection-diffusion form in the continuity equations, but is described through the magnetic force term in the toroidal momentum equation [8]. A main drawback is that the governing equations are built on a circular concentric equilibrium, i.e. the cylindrical coordinates (r, θ, ϕ). In this sense, some physics originating from geometry such as the Pfirsch-Schlüter flux has been dropped. It is, furthermore, natural to construct a momentum equation in the direction parallel to the magnetic field line rather than that in the poloidal direction, because the parallel motion of particles is essential for, especially neoclassical, transport phenomena in toroidal plasmas. Hence we will derive the governing equations of TASK/TX in the $\boldsymbol{b}$, $\nabla\zeta$ and $\nabla\rho$ directions on the axisymmetric flux coordinates (ρ, θ, ζ).

Hereafter we will expand plasma parameters in terms of a small gyroradius, $\delta \equiv \varrho/L \ll 1$, and take into account their non-perturbed (lowest) part solely. Recalling that the flow within a flux surface is $O(\delta)$ and the transport flux across a flux surface is $O(\delta^2)$, we assume that the drift ordering ($\partial/\partial t \sim \delta\omega_t$, $u \sim \delta v_t$) is appropriate for the momentum equations whereas the transport ordering ($\partial/\partial t \sim \delta^2\omega_t,\ u \sim \delta v_t$) for the otherwise equations, where ω_t and v_t denote the transit frequency and the thermal velocity, respectively. Note that in what follows the notation is standard and subscripts denoting the species will be dropped unless otherwise specified.

Straight field-line coordinates and Maxwell's equations

In the straight field-line coordinates, the magnetic field can be written as

$$\boldsymbol{B} = \nabla\psi\times\nabla(q\theta-\zeta) = \nabla\times(\psi_t\nabla\theta-\psi\nabla\zeta) = \nabla\zeta\times\nabla\psi+\nabla\psi_t\times\nabla\theta.$$

Comparing it to $\boldsymbol{B} = \nabla\times\boldsymbol{A}$, we have $\boldsymbol{A} = \psi_t\nabla\theta-\psi\nabla\zeta$ and thus $A_\rho = 0$, $A_\theta = \psi_t$ and $A_\zeta = -\psi$. Here, the subscripts (superscripts) of the coordinates denote the covariant (contravariant) components of a vector. In axisymmetric devices, the partial orthogonality, i.e. $\nabla\rho\cdot\nabla\zeta = 0 = \nabla\theta\cdot\nabla\zeta$, yields

$$\boldsymbol{B} = \nabla\zeta\times\nabla\psi+I\nabla\zeta, \quad \text{where } I = RB_t = \frac{R^2}{\sqrt{g}}\frac{\partial\psi_t}{\partial\rho} = \frac{4\pi^2}{V'\langle R^{-2}\rangle}\frac{\partial\psi_t}{\partial\rho}.$$

Note that $R^2/\sqrt{g}$ is the flux function. $\boldsymbol{B}$ is generally expressed by the contravariant components, such that $B^\rho = \boldsymbol{B}\cdot\nabla\rho = 0$ due to the definition of the magnetic surface.

Maxwell's equations consist of Gauss's law (Poisson's equation), Faraday's law and Ampère's law. From Faraday's law $\boldsymbol{E} = -\nabla\Phi-\dot{\boldsymbol{A}}$, it is found that the electric fields are strongly tied to (the temporal change in) the magnetic fluxes, as follows:

$$E_\theta = \boldsymbol{E}\cdot\sqrt{g}\nabla\zeta\times\nabla\rho = -\frac{\partial A_\theta}{\partial t} = -\frac{\partial\psi_t}{\partial t}, \quad E_\zeta = g_{\zeta\zeta}\boldsymbol{E}\cdot\nabla\zeta = -R^2\frac{\partial A_\zeta}{\partial t} = \frac{\partial\psi}{\partial t}(=RE_t).$$

Ampère's law relates (the spatial gradient of) the magnetic flux to the current. Taking the scalar product of Ampère's law with $\nabla\zeta$ and the subsequent flux-surface average yield

$$\frac{1}{c^2}\frac{\partial\dot{\psi}}{\partial t} = \frac{1}{V'\langle R^{-2}\rangle}\frac{\partial}{\partial\rho}\left[V'\left\langle\frac{|\nabla\rho|^2}{R^2}\right\rangle\frac{\partial\psi}{\partial\rho}\right]-\mu_0\frac{\langle j^\zeta\rangle}{\langle R^{-2}\rangle}, \quad \frac{\partial\psi}{\partial t} \equiv \dot{\psi}(=E_\zeta = RE_t), \tag{1}$$

the latter of which is not only the definition of $\dot{\psi}$ but also one of the governing equations that constitutes Maxwell's equations. This is because it is just another form of Faraday's law. Taking the scalar product of Ampère's law with $\boldsymbol{B}$ and then subtracting (1) give the equations for the toroidal flux in the form:

$$\frac{1}{c^2}\frac{\partial\dot{\psi}_t}{\partial t} = V'\left\langle\frac{|\nabla\rho|^2}{R^2}\right\rangle\frac{\partial}{\partial\rho}\left[\frac{1}{V'\langle R^{-2}\rangle}\frac{\partial\psi_t}{\partial\rho}\right]+\mu_0\frac{\langle Bj_\parallel\rangle-I\langle j^\zeta\rangle}{\langle B^\theta\rangle}, \quad \frac{\partial\psi_t}{\partial t} \equiv \dot{\psi}_t(=-E_\theta). \tag{2}$$

When neglecting the displacement current term that is negligibly small, we readily find that the right-hand sides of (1) and (2) reproduce the expressions of the toroidal and parallel currents. Finally, the Coulomb gauge allows us to simply write flux-surface-averaged Gauss's law as follows:

$$\frac{1}{V'}\frac{\partial}{\partial\rho}\left[V'\langle|\nabla\rho|^2\rangle\frac{\partial\Phi}{\partial\rho}\right] = -\frac{1}{\epsilon_0}\sum_s e_s n_{s0}. \tag{3}$$

Continuity equations

A flux-surface-averaged continuity equation is simply given by

$$\frac{1}{V'}\frac{\partial}{\partial t}(V'n_0)\bigg|_\rho + \frac{1}{V'}\frac{\partial}{\partial \rho}[V'n_0\langle(\boldsymbol{u}-\boldsymbol{u}_g)\cdot\nabla\rho\rangle] = \langle S_{\mathrm{n2}}\rangle. \tag{4}$$

TASK/TX is distinguished from other similar codes in that $n_0\langle(\boldsymbol{u}-\boldsymbol{u}_g)\cdot\nabla\rho\rangle$ is not approximated by the convection-diffusion relationship, but is treated as a dependent variable: The grid velocity $\boldsymbol{u}_g$ and the flux $n_0\langle\boldsymbol{u}\cdot\nabla\rho\rangle$ are both self-consistently calculated in the system.

Momentum equations

The parallel motion of plasma species regulates neoclassical properties in axisymmetric systems. Using the non-conservative form of the momentum equation, we have up to $O(\delta)$

$$mn_0\frac{\partial\langle Bu_\parallel\rangle}{\partial t}\bigg|_\rho = -\langle\boldsymbol{B}\cdot\nabla\cdot\overleftrightarrow{\pi}\rangle + \langle\boldsymbol{B}\cdot\boldsymbol{F}\rangle + en_0\langle BE^A_\parallel\rangle + \langle BS_{\mathrm{m}\parallel}\rangle. \tag{5}$$

Here, we have deliberately left some possibly higher-order terms that may be effective in some cases, such as the source term. The lowest order of the viscosity $\overleftrightarrow{\pi}$ corresponds to the neoclassical viscosity in this parallel equation. Therefore, the viscous term and the friction term can be described in the manner of the moment approach [9]. An unsolved parallel heat flux, which means that a heat flux is not a dependent variable of TASK/TX, can be expressed in the form: $\boldsymbol{q} = (\overleftrightarrow{L}_{22} - \overleftrightarrow{M}_3)^{-1}\left[(\overleftrightarrow{L}_{21} + \overleftrightarrow{M}_2)\boldsymbol{u} - \boldsymbol{D}\right]$, where $\overleftrightarrow{L}$, $\overleftrightarrow{M}$, $\boldsymbol{u}$ and $\boldsymbol{D}$ denote the friction and viscosity matrices and the parallel and diamagnetic flows, respectively. The matrices can be computed by the Matrix Inversion [10], for example. When substituting the heat flux into (5), we have

$$\begin{aligned}
m_a n_{a0}\frac{\partial\langle Bu_{a\parallel}\rangle}{\partial t}\bigg|_\rho &= \left[-\hat{\mu}^a_1 + \ell^{aa}_{11} - (\hat{\mu}^a_2 + \ell^{aa}_{12})(\mathrm{A}^{-1}\mathrm{B})_a\right]\langle Bu_{a\parallel}\rangle + \sum_{b\neq a}\left[\ell^{ab}_{11} - \ell^{ab}_{12}(\mathrm{A}^{-1}\mathrm{B})_b\right]\langle Bu_{b\parallel}\rangle \\
&+ \sum_b\left[\delta_{ba}\{\hat{\mu}^b_1 + (\hat{\mu}^b_2)^2(\mathrm{A}^{-1})_b\} + \ell^{ab}_{12}\hat{\mu}^b_2(\mathrm{A}^{-1})_b\right]\langle BV_{1b}\rangle \\
&+ \sum_b\left[\delta_{ba}\{\hat{\mu}^b_2 + \hat{\mu}^b_2\hat{\mu}^b_3(\mathrm{A}^{-1})_b\} + \ell^{ab}_{12}\hat{\mu}^b_3(\mathrm{A}^{-1})_b\right]\langle BV_{2a}\rangle + e_a n_{a0}\langle BE^A_\parallel\rangle + \langle BS^{\,a}_{\mathrm{m}\parallel}\rangle,
\end{aligned} \tag{6}$$

where δ_{ba} denotes the Kronecker delta, $\mathrm{A} \equiv \overleftrightarrow{L}_{22} - \overleftrightarrow{M}_3$, and $\mathrm{B} \equiv \overleftrightarrow{L}_{21} - \overleftrightarrow{M}_2$.

The toroidal momentum equation is important for TASK/TX in that it governs not only radial transport of the toroidal momentum but also provokes a particle flux as well as a $\boldsymbol{j}\times\boldsymbol{B}$ torque through the magnetic force term. As seen in [7, 9], the collisional, i.e. classical and neoclassical, particle flux is theoretically derived by the toroidal momentum equation. In conventional transport codes the particle transport coefficients that are computed by external modules (cf. [6, 10]) are directly substituted in the particle transport equation, whereas in TASK/TX coupling of the parallel and toroidal momentum equations and the continuity equations self-consistently brings about the particle flux. Using the conservative form of the momentum equation, we have up to $O(\delta^2)$

$$\frac{1}{V'}\frac{\partial}{\partial t}(V' m n_0 \langle R u_\zeta\rangle) = -\langle R^2\nabla\zeta\cdot\nabla\cdot\overleftrightarrow{\pi}\rangle + \langle R F_\zeta\rangle + e n_0\langle R E^A_\zeta\rangle + e n_0\frac{\partial\psi}{\partial\rho}\langle \boldsymbol{u}\cdot\nabla\rho\rangle + \langle R S_{\mathrm{m}\zeta 2}\rangle \quad (7)$$

While the first-order viscous stress term vanishes due to the CGL form, the second-order term can be expressed as a combination of a convective momentum flux, a.k.a. inward pinch, and a diffusive one plus a residual stress. The friction term can be expressed in the similar manner as shown above. Furthermore, we have to add a turbulent force F_a^{QL} that drives a turbulence-induced quasilinear particle flux [8]. The toroidal momentum equation is finally given by

$$\frac{1}{V'}\frac{\partial}{\partial t}(V' m_a n_{a0}\langle R u_{a\zeta}\rangle) = -\frac{1}{V'}\frac{\partial}{\partial\rho}V'\left[\langle|\nabla\rho|\rangle v_a m_a n_{a0}\langle R u_{a\zeta}\rangle - \langle|\nabla\rho|^2\rangle\chi_{a\zeta} m_a n_{a0}\frac{\partial\langle R u_{a\zeta}\rangle}{\partial\rho} + \langle\Pi_a^{\mathrm{res}}\rangle\right]$$
$$+\ell_{11}^{aa}\langle R u_{a\zeta}\rangle + \sum_{b\neq a}\ell_{11}^{ab}\langle R u_{b\zeta}\rangle - \frac{I}{\langle B^2\rangle}\sum_b \ell_{12}^{ab}\Big[(\mathrm{A}^{-1}\mathrm{B})_b\langle B u_{b\|1}\rangle - (\mathrm{A}^{-1})_b\hat{\mu}_2^b\langle B V_{1b}\rangle$$
$$+\left\{-(\mathrm{A}^{-1})_b\hat{\mu}_3^b I + \left(\langle R^2\rangle\langle B^2\rangle - I^2\right)\right\}\frac{\langle B V_{2b}\rangle}{I}\Big] + e_a n_{a0}\langle R E_\zeta^A\rangle + e_a n_{a0}\frac{\partial\psi}{\partial\rho}\langle\boldsymbol{u}_a\cdot\nabla\rho\rangle + F_a^{\mathrm{QL}}. \quad (8)$$

The radial momentum equation or the radial force balance equation is essentially identical to the first-order flow within the flux surface. The leading order is $O(1)$ for pressure and Lorentz force terms, and the other terms are practically ineffective. It is useful to introduce the incompressibility of the flow when we average the equation over the flux surface. Thus, we obtain

$$0 = -\frac{\partial p_{a0}}{\partial\psi}(\langle B^2\rangle\langle R^2\rangle - I^2) - e_a n_{a0}\frac{\partial\Phi}{\partial\psi}(\langle B^2\rangle\langle R^2\rangle - I^2) + e_a n_{a0} I\langle B u_{a\|}\rangle - e_a n_{a0}\langle B^2\rangle\langle R u_{a\zeta}\rangle. \quad (9)$$

In summary regarding the momentum equations, the dependent variables are $\langle B u_{a\|}\rangle$, $\langle R u_{a\zeta}\rangle$ and $\langle\boldsymbol{u}_a\cdot\nabla\rho\rangle$. A set of the governing equations is completed after further deriving the equations for heat and beam components.

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