MHD turbulence simulations with discontinuous Galerkin methods

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Introduction

The special relativistic magnetohydrodynamics equations model a wide variety of physical phenomena in particle physics as well as in astrophysics. They are quite used in relativistic astrophysics where high energy and intense magnetic fields effects play a very important role. Typical examples in this scenario are relativistic jets in young stellar objects and extragalactic radio sources, accretion flows around massive compact objects, pulsar winds and gamma ray bursts. Due to the high complexity of this equations, solving them analytically is only possible in a few particular and very simple cases. Therefore it is necessary to construct numerical methods to solve them and to analyze correctly astrophysical phenomena.

In computational fluid dynamics, high order numerical methods have gained quite popularity in the last years due to the need of high fidelity predictions in the simulations. Among these methods, the family of Discontinuous Galerkin (DG) schemes are in discussion as future solvers in hydrodynamic flow problems because of their excellent properties and efficiency for complex flows and geometries [2, 5]. So far, the employment of DG schemes in numerical simulations have been mainly concentrated in the solution of the Euler and Navier-Stokes equations as well as the Maxwell equations, i.e. for engineering applications. Our objective then is to construct very high order numerical methods of Discontinuous Galerkin Spectral Element type for the special relativistic magnetohydrodynamics equations in multiple space dimensions on structured hexahedral meshes, as we believe that simulating the fluid flow and shock wave patterns very efficiently and with a very high accuracy will help to analyze the processes at work in these astrophysical plasmas with much more precision.

In this work we will show numerical results of one of the standard test problems for the SRMHD equations in 2D, the relativistic version of the compressible Orszag-Tang vortex, which represents a good example of MHD driven turbulence development.

Relativistic Magnetohydrodynamics Equations

As we mentioned above, our goal is to solve numerically the special relativistic MHD equations. A fundamental aspect of the DGSEM methods is to write the PDE in conservation form.

Therefore we write the SRMHD equations as follows

$$\frac{\partial \mathbf{U}}{\partial t} + \sum_{i=1}^{3} \frac{\partial \mathbf{F}^i(\mathbf{U})}{\partial x^i} = 0, \tag{1}$$

where the state vector $\mathbf{U}$ and the flux vectors $\mathbf{F}^i$ are given by

$$\mathbf{U} = \begin{pmatrix} D \\ S^1 \\ S^2 \\ S^3 \\ B^1 \\ B^2 \\ B^3 \\ E \end{pmatrix}, \quad \mathbf{F}^i = \begin{pmatrix} Dv^i \\ S^1 v^i - B_i\left(B_1/\Gamma^2 + (\mathbf{v}\cdot\mathbf{B})v^1\right) + p\delta^{1i} \\ S^2 v^i - B_i\left(B_2/\Gamma^2 + (\mathbf{v}\cdot\mathbf{B})v^2\right) + p\delta^{2i} \\ S^3 v^i - B_i\left(B_3/\Gamma^2 + (\mathbf{v}\cdot\mathbf{B})v^3\right) + p\delta^{3i} \\ B^1 v^i - B^i v^1 \\ B^2 v^i - B^i v^2 \\ B^3 v^i - B^i v^3 \\ S^i \end{pmatrix}. \tag{2}$$

Here, $p = p_g + |\mathbf{B}|^2/(2\Gamma^2) + (\mathbf{v}\cdot\mathbf{B})^2/2$ is the total pressure, p_g is the thermal (gas) pressure and $\mathbf{v} = (v^1, v^2, v^3)$ is the fluid three-velocity.

$$D = \rho\Gamma, \tag{3}$$

$$S^i = \left(\rho h \Gamma^2 + |\mathbf{B}|^2\right) v^i - (\mathbf{v}\cdot\mathbf{B})B^i, \tag{4}$$

$$E = \rho h \Gamma^2 - p_g + \frac{|\mathbf{B}|^2}{2} + \frac{|\mathbf{v}|^2 |\mathbf{B}|^2 - (\mathbf{v}\cdot\mathbf{B})^2}{2}, \tag{5}$$

where v^i are the components of the three-velocity of the fluid, Γ is the Lorentz factor,

$$\Gamma = \frac{1}{\sqrt{1 - v^i v_i}}, \tag{6}$$

ρ is the proper mass density and h is the specific enthalpy. This system of partial differential equations is closed with an equation of state $h = h(p, \rho)$. In the book of Anile [1], it is shown that this system is hyperbolic for causal equations of state, i.e., for those where the local sound speed satisfies $c_s < 1$, where c_s is defined by

$$hc_s^2 = -\frac{\rho}{\kappa h}\frac{\partial h}{\partial \rho}, \quad \kappa = \rho\frac{\partial h}{\partial p} - 1. \tag{7}$$

Discontinuous Galerkin Spectral Element Framework

Here we will describe in brief the main features of the DGSEM framework. For more details we suggest to see the paper of Hindenlang et al. [4] and for the full implementation, please see the book of Kopriva [5]. We start by subdividing the physical domain into hexahedral elements. Then, we apply a mapping to each one of these elements onto the reference unit cube element

$[-1,1] \times [-1,1] \times [-1,1]$. Next, the partial differential equations of the SRMHD written in conservation form are mapped into the reference element space. We apply the following ansatz for the vector of conservatives variables and the physical fluxes: they are approximated by a tensor-product basis of 1D Lagrange interpolating polynomials. Following the methodology of the standard discontinuous Galerkin method, we multiply with a test function the transformed PDE and integrate over the reference element. We get the weak form of the PDE, where the volume and surface integrals are replaced by Gauss-Legendre quadrature rules. In the surface integrals, the fluxes are calculated with an approximated Riemann solver. In this work were employed the Rusanov flux, the HLLC Riemann solver [6] and the HLLD Riemann solver for relativistic MHD. Runge-Kutta methods of fourth order of accuracy were used to evolve the equations in time. Since the numerical solution of the SRMHD equations have to satisfy the solenoidal constraint $\nabla \cdot \mathbf{B} = 0$, we have chosen the Generalized Lagrange Multiplier hyperbolic transport correction of Dedner et al. [3], but with the speed of propagation of the local divergence errors limited to the speed of light. Due to the presence of shocks in high energy astrophysical phenomena, TVD limiters are required.

Numerical Calculations

Now we show results of the simulation of the Orszag-Tang problem. The Orszag-Tang Vortex is a well-known test for MHD codes, but it can be easily extended to the relativistic MHD equations. In this problem, the initial conditions lead to a system of supersonic MHD driven turbulence. Therefore, this problem represents a good test for our code if we want that it be able to handle turbulence and shocks. The computational domain is the square $[0,2\pi] \times [0,2\pi]$, which was subdivided in 32×32 cells. The boundary conditions are periodic in all faces of the square and the initial state is given by

$$(\rho, v_x, v_y, p, B_x, B_y) = (\frac{25}{9}, -\sin(y), \sin(x), \frac{5}{3}, -\sin(y), \sin(2x)) \tag{8}$$

The adiabatic index is taken as $\gamma = 1.6667$, and the end time of the simulation is $t_f = 3.0$. The time discretization was performed by a Runge-Kutta method of fourth order of accuracy and the polynomial degree was $N = 3$. In Figure 1 are shown the B_x and B_y at time $t = 3.0$.

Summary

The Discontinuous Galerkin Spectral Element framework has been successfully implemented for the solution of the relativistic magnetohydrodynamics. The Orszag-Tang vortex was calculated and it allow us to conclude that our code based on DGSEM methods is able to handle shocks as well as turbulence. In fact, we could see the development of turbulence and the appearance and interaction of shocks within the physical domain. Further work includes the simulation

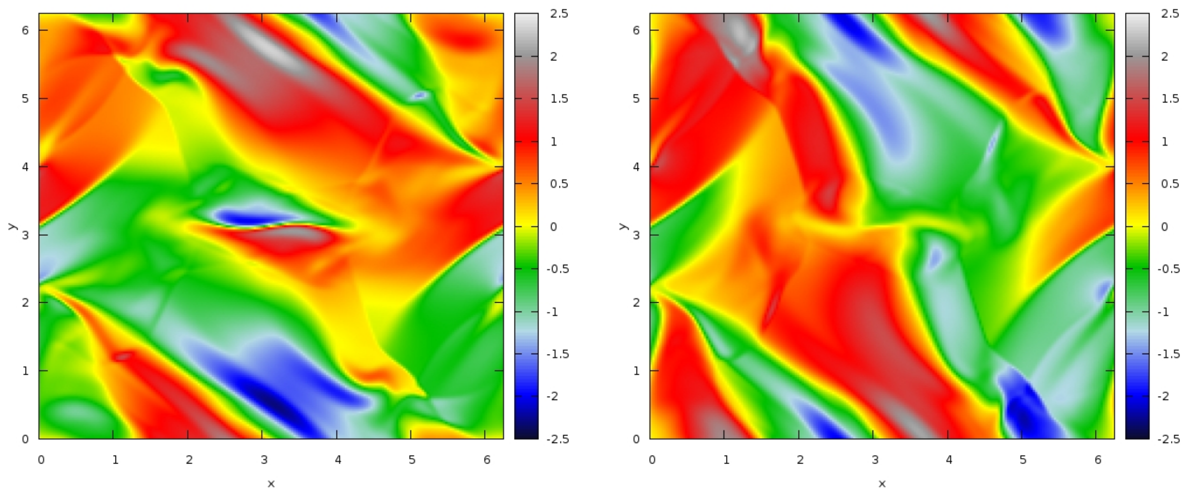

Figure 1: The relativistic version of the Orszag-Tang vortex at simulation time $t = 3.0$. From left to right, B_x, and B_y. Calculation done with a polynomial degree $N = 3$ on a mesh of 32×32 cells.

of relativistic jets with different configurations of the magnetic field, where are present, in many cases, MHD driven turbulence, like the Kelvin-Helmholtz instabilities.

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