Application of Picard iteration technique to self-consistent wave-particle interaction in plasmas

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Abstract The validity of quasilinear theory to describe the weak warm beam-plasma instability is still an open issue. This work extends the analytical approach of the problem with a model where the beam is described as a set of particles while the waves are harmonic oscillators. An average over the initial wave phases is performed over the result of the third iterate of the Picard iteration technique applied to the equations of motion. This calculation shows that some of the results of the uniform particle density case remain correct even for a non uniform density. However, as for Langmuir wave amplitude evolution, there is a spontaneous emission by spatial inhomogeneities (turbulent eddies) on top of Landau growth or damping and of spontaneous emission by particles.

The validity of quasilinear theory (QL) describing the weak warm beam-plasma instability has been a controversial topic for several decades (see the many references in the introduction of [1]). It involves the chaotic dynamics of self-consistent wave-particle dynamics, whose description sounds formidable for a Vlasovian description. This was an incentive to tackle this instability by generalizing [2, 3] a model originally introduced for the numerical simulation of the cold beam-plasma instability [4, 5]. This model describes the beam as a set of particles, while the waves are present as harmonic oscillators. A Langmuir wave with a phase velocity ω/k in a range of velocities where there are no resonant particles verifies the Bohm-Gross dispersion relation, and is equivalent to a harmonic oscillator. Wave-particle dynamics is described by the self-consistent Hamiltonian

$$H_{\rm sc} = \sum_{r=1}^{N} \frac{p_r^2}{2} + \sum_{j=1}^{M} \omega_{j0} I_j - \varepsilon \sum_{r=1}^{N} \sum_{j=1}^{M} k_j^{-1} \beta_j \sqrt{2I_j} \cos(k_j x_r - \theta_j) \tag{1}$$

where $\varepsilon = \omega_{\rm p}[2m\eta/N]^{1/2}$ is the coupling parameter and $\beta_j = [\partial\varepsilon_{\rm d}(k_j,\omega_{j0})/\partial\omega]^{-1/2}$, with $\omega_{\rm p}$ the plasma frequency, m the mass of particles set to unity, η the ratio of the tail to the bulk density, $\varepsilon_{\rm d}(k,\omega)$ the bulk dielectric function, k_j and ω_{j0} the wavenumber and pulsation of wave j. The conjugate variables for $H_{\rm sc}$ are (p_r, x_r) for the particles and (I_j, θ_j) for the waves. On top of the total energy $E_{\rm sc} = H_{\rm sc}$, the total momentum $P_{\rm sc} = \sum_{r=1}^{N} p_r + \sum_{j=1}^{M} k_j I_j$ is conserved.

This model was derived from the N-body description of the beam-plasma system [6]. More recently, this was done again in a heuristic way (see section 2.1 of [7]), and in a rigorous one by a series of controlled approximations (see the remaining of chapter 2 of [7]), which enables replacing the many particles of the bulk by their collective vibrations. This approach provided a rigorous mechanical understanding of Landau growth and damping, and helped into the investigation of the nonlinear evolution of the weak warm beam-plasma instability. In particular, it proved that the wave spectrum is almost frozen when the plateau is formed in the particle distribution function [1].

In order to find out the nature of the wave spectrum at saturation, numerical simulations were performed using a semi-Lagrangian code for the Vlasov–wave model [1]. This model is the mean-field limit of the granular dynamics defined by the self-consistent Hamiltonian: waves are still present as M harmonic oscillators, but particles are described by a continuous distribution function. The simulations were repeated for a large number of random realizations of the initial wave phases for a fixed initial spectrum of amplitudes, and showed the plateau verified the predictions of QL theory. They also brought an unexpected result: the variation of the phase of a given wave with time was found to be almost non fluctuating with the random realizations of the initial wave phases [8].

This suggested revisiting the past analytical calculations of the wave phase and amplitude average evolutions [7, 9] which were performed by averaging over the initial particle positions. The new numerical result [8] suggested to perform instead an average over the initial wave phases, which is compatible with a non uniform particle density. Furthermore, the previous calculations used a perturbative approach which made sense in the linear regime, but which are a priori unjustified for the chaotic regime of the instability. This was an incentive to use the Picard iteration technique which is the central tool to prove the existence and uniqueness of solutions to differential equations in the so-called Picard's existence theorem, Picard-Lindelöf theorem, or Cauchy-Lipschitz theorem. Picard iteration technique considers the solution of differential equation $\frac{\mathrm{d}\mathbf{X}}{\mathrm{d}t} = f(\mathbf{X},t)$, where $\mathbf{X}$ is an N-dimensional vector, as the fixed point of the mapping $\mathbf{X}_n \mapsto \mathbf{X}_{n+1}$, $n = 0,1,...$ defined by $\frac{\mathrm{d}\mathbf{X}_{n+1}}{\mathrm{d}t} = f(\mathbf{X}_n,t)$, and by some approximation $\mathbf{X}_0$ of the solution.

The iteration turns out to be analytically tractable three times when starting from the ballistic approximation of the dynamics. It can be shown analytically that the third order Picard iterate is able to describe the separation between trapped and passing orbits of a nonlinear pendulum. Furthermore, numerical calculations [10] indicate that, for the chaotic motion of particles in a prescribed set of waves, such an iterated solution is already fairly good over the Lyapunov timescale which plays the role of a decorrelation time. However the accuracy of the result obtained with the third order Picard iterate needs further assessment.

This work in progress already brings the following results. First, the modification of the average wave frequency due to the coupling with particles is exactly the principal part correction to the wave frequency provided by the Vlasovian calculation of the dispersion relation of Langmuir waves or by the equivalent calculations with the self-consistent Hamiltonian [7]. However, the latter calculations deal with a spatially uniform distribution of particles, while the present one holds whatever the spatial inhomogeneity of the distribution of tail particles, but requires an average over the phases of the Langmuir waves. Second, an estimate of phase fluctuations shows they scale like $\eta^{1/2}$, which makes them negligible, as shown by simulations. Therefore, if initial phases are random, they stay random for all times: there is no need for the traditional random phase approximation.

Let $\tau_{\mathrm{D}} = (k^2 D_{\mathrm{QL}})^{-1/3}$ be the Dupree timescale, where k is the typical value of k_j and D_{QL} is the quasilinear diffusion coefficient which scales like the typical wave amplitude squared. Third, assuming the wave spectrum of any realization to be smooth when averaged over a width in phase velocity on the order of $(k\tau_{\mathrm{D}})^{-1}$, the evolution of a wave amplitude A is given by

$$\frac{\mathrm{d}\langle |A_j|^2\rangle}{\mathrm{d}t} = 2\gamma_{j\mathrm{L}}\langle |A_j|^2\rangle + S_{\mathrm{spont}j} + S_{\mathrm{inhom}j}, \tag{2}$$

where

$$\gamma_{j\mathrm{L}} = \alpha_j \frac{\mathrm{d}\bar{f}}{\mathrm{d}v}(\omega_{j0}/k_j) \tag{3}$$

with $\alpha_j = \frac{\pi}{2} N \varepsilon_j^2 k_j^{-2}$ which is finite in the the limit $N \to \infty$, and where $\bar{f}(v)$ is the space averaged coarse-grained velocity distribution function of the tail particles. Equation (2) displays successively the contribution to the wave amplitude evolution of Landau growth or damping, of spontaneous emission, and of the emission of spatial inhomogeneities (turbulent eddies). It contains the spontaneous emission term of the previous calculations averaging over particle positions [7, 9], $S_{\mathrm{spont}j} \sim \bar{f}(\omega_{j0}/k_j)/N$. A new term of emission by turbulent eddies shows up

$$S_{\mathrm{inhom}j} = \frac{N^2 \varepsilon_j^4}{k_j^2} \int_0^t \mathrm{d}t' \int\int \mathrm{d}p\mathrm{d}p' \mathrm{e}^{\mathrm{i}[\Omega_j(p')t' - \Omega_j(p)t]} \langle \tilde{f}(-k_j, p', t') \tilde{f}(k_j, p, t)\rangle + \mathrm{c.c.}. \tag{4}$$

where k is the wavenumber, $\tilde{f}$ is the Fourier transform of the coarse-grained velocity distribution function, $\Omega_j(p) = k_j p - \omega_{j0}$, and t is on the order of τ_{D}.

Because of the $1/N$ factor, spontaneous emission vanishes when $N \to \infty$, since plasma graininess becomes negligible. To the contrary, the contribution of inhomogeneities to wave emission does not vanish in this limit. If $f(x,v)$, the coarse-grained velocity distribution function of the tail particles, does not depend on x, $S_{\mathrm{inhom}j}$ vanishes. This occurs in particular when the plateau forms at the end of the weak beam-plasma instability in the limit $N \to \infty$ [1]. If such an instability starts from a position-independent velocity distribution function, the $\tilde{f}$'s are only due to turbulent eddies. Then the size of $S_{\mathrm{inhom}j}$ can be bounded by a quantity vanishing in the limit where the number of waves is large, i.e. for a continuous wave spectrum. Therefore, if these calculations make sense, in this limit the quasilinear equations might correctly describe the average behavior of the instability, even though a given realization be very far away from the average behavior.

The spontaneous emission by turbulent eddies can be understood as follows. For a given realization, trapping motion is observed in phase space, which produces turbulent eddies with a range in velocity $\Delta v_{\mathrm{D}} = (k\tau_{\mathrm{D}})^{-1}$. For simplicity, let us take $\omega_{j0} = \omega_{\mathrm{p}} = 1$. Then the phase velocity of a wave with wavenumber k_0 is $v = 1/k_0$. Therefore a width Δv_{D} in phase velocity corresponds to a width in k which is $\Delta k = k^2 \Delta v_{\mathrm{D}}$. This in turn defines the typical spatial width of eddies $\Delta x_{\mathrm{ed}} = (k^2 \Delta v_{\mathrm{D}})^{-1}$. Imagine at a given time the velocity distribution is spatially uniform over a given eddy, but has a finite slope in velocity. Then a quarter of trapping period in the eddy later, this finite slope in velocity turns into a finite slope in position, i.e. into a spatial inhomogeneity over Δx_{ed}. During the growth of the beam-plasma instability, this fluctuating spatial inhomogeneity process sets in all over phase space. This graininess yields spontaneous emission by turbulent eddies.

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