KdV equations for three-dimensional dust-lattice waves

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The Coulomb dusty plasmas are a finite number of charged micro-particles interacting via a shielded strong Coulomb potential and confined by external forces. The interaction potential is believed to be of the Yukawa type. The formation of ordered arrangements of dust particles in a gas discharge plasma was predicted by Ikezi [1] and such "plasma crystals" were experimentally realized in 1994 [2]. Various two and three dimensional crystalline configurations are observed in experiments i.e. 2D hexagonal, 3D body-centered cubic (bcc) and face-centered cubic (fcc) [3]. Wang *et.al.*[4] has studied the linear and nonlinear waves in (3D) simple cubic configuration. The experiments show that the Yukawa systems have a little chance to be in simple cubic configuration, so we have followed the other configurations. The state of an infinite system of particles with Yukawa pair interaction is determined by the coupling strength between the particles (parameter Γ is measured in units of the potential energy of interaction between neighboring particles normalized by their mean kinetic energy) and the lattice parameter κ. For $\Gamma \sim 10^3$, $\kappa \sim 1-2.5$, the state of particles can crystallize into bcc configuration.

In this paper, we consider nonlinear dust lattice waves propagating in 3D dusty plasma crystal (bcc configuration). Here we keep interaction between particles up to the second neighbor and nonlinearity due to the geometry and expansion of Yukawa potential energy. We obtain the KdV equation for longitudinal dust-lattice wave propagating along principal directions in lattice.

Consider a three-dimensional cubic crystal with lattice spacing a consisting of negative charge dust grains q and mass M, modeled as point charges. We may expand the potential energy around equilibrium at $r = r_0$,

$$U(r) \cong \frac{1}{2}G_1(r-r_0)^2 + \frac{1}{3}G_2(r-r_0)^3 + \frac{1}{4}G_3(r-r_0)^4 + \cdots, \tag{1}$$

where $(\partial U/\partial r)|_{r=r_0} = 0$, $G_1 = (\partial^2 U/\partial r^2)|_{r=r_0}$, $G_2 = (\partial^3 U/\partial r^3)|_{r=r_0}/2$, and $G_3 = (\partial^4 U/\partial r^4)|_{r=r_0}/6$. The bcc configuration has eight nearest neighbors ($r_0 = a\sqrt{3}/2$) with the spring constant G_1, G_2, G_3 and six second neighbors ($r_0 = a$) with the spring constant G_1', G_2', G_3'.

$$G_1 = \frac{q^2\exp(-\kappa\sqrt{3}/2)}{4\pi\varepsilon_0\lambda_D^3}\frac{16+8\sqrt{3}\kappa+6\kappa^2}{3\sqrt{3}\kappa^3}, \qquad G_1' = \frac{q^2\exp(-\kappa)}{4\pi\varepsilon_0\lambda_D^3}\frac{2+2\kappa+\kappa^2}{\kappa^3} \tag{2}$$

$$G_2 = -\frac{q^2\exp(-\kappa\sqrt{3}/2)}{4\pi\varepsilon_0\lambda_D^4}\frac{16+8\sqrt{3}\kappa+6\kappa^2+\sqrt{3}\kappa^3}{3\kappa^4}\exp(-\kappa\sqrt{3}/2), \qquad G_2' = -\frac{q^2\exp(-\kappa)}{4\pi\varepsilon_0\lambda_D^4}\frac{6+6\kappa+3\kappa^2+\kappa^3}{2\kappa^4} \tag{3}$$

$$G_3 = \frac{q^2 \exp(-\kappa\sqrt{3}/2)}{4\pi\varepsilon_0\lambda_D^5}\frac{128+64\sqrt{3}\kappa+48\kappa^2+8\sqrt{3}\kappa^3+3\kappa^4}{9\sqrt{3}\kappa^5}, \quad G_3' = \frac{q^2\exp(-\kappa)}{4\pi\varepsilon_0\lambda_D^5}\frac{24+24\kappa+12\kappa^2+4\kappa^3+\kappa^4}{6\kappa^5} \tag{4}$$

where the lattice parameter $\kappa = a/\lambda_D$ is a positive real number, practically taking values $\kappa \cong 1-2.5$ in real discharge experiments.

Let the (n,m,l)th particle location define the origin, then the positions of the first neighbors at equilibrium are $a(\pm 1,\pm 1,\pm 1)/2$. However, if the particles are not at their equilibrium positions, we then define the eight lengths to represent the distances from particle (n,m,l) to the nearest particles, respectively,

$$r_{i1,i2,i3} = \sqrt{(i1+\Delta u_{i1,i2,i3})^2 + (i2+\Delta v_{i1,i2,i3})^2 + (i3+\Delta w_{i1,i2,i3})^2}, \tag{5}$$

where $\Delta u_{i1,i2,i3} = u_{m+i1,n+i2,l+i3} - u_{mnl}$, $\Delta v_{i1,i2,i3} = v_{m+i1,n+i2,l+i3} - v_{mnl}$, $\Delta w_{i1,i2,i3} = w_{m+i1,n+i2,l+i3} - w_{mnl}$, ($i1$, $i2$, $i3$) are $a(\pm 1,\pm 1,\pm 1)/2$ for different neighbors. For the second neighbors, we define six lengths to represent the distances from particle (n,m,l) and ($i'1$, $i'2$, $i'3$) are $a(\pm 1,0,0)$, $a(0,\pm 1,0)$, $a(0,0,\pm 1)$ for different neighbors.

In the case of propagating the longitudinal wave in (1,0,0)-direction, $\Delta v = \Delta w = 0$, so the equation of motion for (nml)th particle in the lattice is

$$\begin{aligned} M\frac{d^2 u_{nml}}{dt^2} &= aG_1 \sum_{i1,i2,i3} [9(\Delta u_{i1,i2,i3}/a) + 18 sign(i1)(\Delta u_{i1,i2,i3}/a)^2 - 8(\Delta u_{i1,i2,i3}/a)^3]/27 \\ &+ G_1'[\Delta u_{m+1,n,l} + \Delta u_{m-1,n,l}] + a^2\sqrt{3}G_2 \sum_{i1,i2,i3}[3 sign(i1)(\Delta u_{i1,i2,i3}/a)^2 + 8(\Delta u_{i1,i2,i3}/a)^3]/27 \\ &+ G_2'[(\Delta u_{m+1,n,l})^2 - (\Delta u_{m-1,n,l})^2] + G_3 \sum_{i1,i2,i3}[(\Delta u_{i1,i2,i3}/a)^3]/9 + G_3'[(\Delta u_{m+1,n,l})^3 + (\Delta u_{m-1,n,l})^3] \end{aligned}. \tag{6}$$

To solve Eq. (6) in the continuum limit, i.e. assuming a weak variation of the function's value in space, we shall consider the displacements to be a continuous function of the space coordinate x, viz. $u_{nml} \to u(x,t)$. We may therefore use the following expansions (this is essentially a simple Taylor expansion around an equilibrium spacing r_0),

$$u_{n\pm 1,m\pm 1,l\pm 1} \approx u_x \pm r_0 u_x + r_0^2 u_{xx}/2 \pm r_0^3 u_{xxx}/6 + r_0^4 u_{xxxx}/24 \pm \ldots. \tag{7}$$

Substituting (7) into Eq. (6), and neglecting terms of order higher than a^4, the classical Newton's law take the form of differential equation for the particle displacements.

$$\begin{aligned} \frac{d^2u}{dt^2} = a^4 \Bigg\{ &\frac{(\omega_{01}^2 + 3\omega_{01}'^2)}{3a^2}\frac{\partial^2 u}{\partial x^2} + \frac{(\omega_{01}^2 + 12\omega_{01}'^2)}{144}\frac{\partial^4 u}{\partial x^4} + \frac{(6\omega_{01}^2 + a\sqrt{3}\omega_{02}^2 + 18a\omega_{02}'^2)}{9a^2}\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} \\ &+ \frac{(-8\omega_{01}^2 + a72\sqrt{3}\omega_{02}^2 + 3a^2\omega_{03}^2 + 108a^2\omega_{03}'^2)}{36a^2}\left(\frac{\partial u}{\partial x}\right)^2\frac{\partial^2 u}{\partial x^2}\Bigg\} \end{aligned} \tag{8}$$

where $\omega_{0i}^2 = G_i/M$. By the same calculations for the longitudinal wave in (1,1,0)-direction ($\Delta w = 0$, $\Delta u = \Delta v$), and (1, 1, 1)-direction($\Delta u = \Delta v = \Delta w$), one can obtain the continuous equation of motions for both directions, respectively.

$$\frac{d^2u}{dt^2}=a^4\left\{\frac{(4\omega_{01}^2+3\omega_{01}'^2)}{6a^2}\frac{\partial^2u}{\partial x^2}+\frac{(4\omega_{01}^2+3\omega_{01}'^2)}{144}\frac{\partial^4u}{\partial x^4}+\frac{(24\omega_{01}^2+27\omega_{01}'^2+16a\sqrt{3}\omega_{02}^2+18a\omega_{02}'^2)}{9a^2}\right.$$
$$\left.\times\frac{\partial u}{\partial x}\frac{\partial^2u}{\partial x^2}+\frac{(-16\omega_{01}^2-81\omega_{01}'^2+64a\sqrt{3}\omega_{02}^2-54a\omega_{02}'^2+96a^2\omega_{03}^2+54a^2\omega_{03}'^2)}{144a^2}\left(\frac{\partial u}{\partial x}\right)^2\frac{\partial^2u}{\partial x^2}\right\} \tag{9}$$

$$\frac{d^2u}{dt^2}=a^4\left\{\frac{(7\omega_{01}^2+3\omega_{01}'^2)}{3a^2}\frac{\partial^2u}{\partial x^2}+\frac{(61\omega_{01}^2+12\omega_{01}'^2)}{144}\frac{\partial^4u}{\partial x^4}+\frac{(24\omega_{01}^2+81\omega_{01}'^2+243a\sqrt{3}\omega_{02}^2+27a\omega_{02}'^2)}{36a^2}\right.$$
$$\left.\times\frac{\partial u}{\partial x}\frac{\partial^2u}{\partial x^2}+\frac{(-16\omega_{01}^2-81\omega_{01}'^2+64a\sqrt{3}\omega_{02}^2-54a\omega_{02}'^2+96a^2\omega_{03}^2+54a^2\omega_{03}'^2)}{144a^2}\left(\frac{\partial u}{\partial x}\right)^2\frac{\partial^2u}{\partial x^2}\right\} \tag{10}$$

Note, we have turned the x-axis towards the wave propagating and have kept displacement $u(x,t)$ in new coordinates. Keeping only the linear contributions in the equations (8)-(10), and assuming a harmonic solution of the type $u\sim\exp[i(kx-\omega t)]+c.c.$, we obtain the dispersion relation in continuous approximation, respectively

$$\omega^2=k^2a^2(\omega_{01}^2+3\omega_{01}'^2)/3-k^4a^4(\omega_{01}^2+12\omega_{01}'^2)/144, \tag{11}$$

$$\omega^2=k^2a^2(4\omega_{01}^2+3\omega_{01}'^2)(24-k^2a^2)/144, \tag{12}$$

$$\omega^2=k^2a^2(7\omega_{01}^2+3\omega_{01}'^2)/3-k^4a^4(61\omega_{01}^2+12\omega_{01}'^2)/144. \tag{13}$$

To study the propagation of small amplitude nonlinear longitudinal dust lattice wave in (1,0,0)-direction, the stretched coordinates are introduced, $\xi=\varepsilon(x-Vt)$, $\tau=\varepsilon^3 t$, where ε is small parameter. Using the reductive perturbation technique in the quasi-continuum limit, the dependent variable $u(x,t)$ is expanded in power of ε ($u=\varepsilon u_1+\varepsilon^2u_2+\cdots$). Substituting this expansion in (8)-(10) and considering the coefficients of each power of ε, at the first order, we obtain the velocity, $V_1=a\sqrt{(\omega_{01}^2+3\omega_{01}'^2)/3}$, $V_2=a\sqrt{(4\omega_{01}^2+3\omega_{01}'^2)/6}$, $V_3=a\sqrt{(7\omega_{01}^2+3\omega_{01}'^2)/3}$ for (1,0,0), (1,1,0), (1,1,1) directions, respectively. In the third order of ε, we obtain the following KdV equation

$$\frac{\partial}{\partial\xi}\left[\frac{\partial U}{\partial\tau}+P_jU\frac{\partial U}{\partial\xi}+Q_j\frac{\partial^3U}{\partial\xi^3}\right]=0, \tag{14}$$

where $U=u_1$ is the longitudinal wave first-term amplitude. Coefficients P_j, Q_j appearing in equation (14) for different directions are given by

(1,0,0): $P_1=\dfrac{(6\omega_{01}^2+a\sqrt{3}\omega_{02}^2+18a\omega_{02}'^2)}{9a^2V_1}$, $Q_1=\dfrac{(\omega_{01}^2+12\omega_{01}'^2)}{144V_1}$,

(1,1,0): $P_2=\dfrac{(24\omega_{01}^2+27\omega_{01}'^2+16a\sqrt{3}\omega_{02}^2+18a\omega_{02}'^2)}{9a^2V_2}$, $Q_2=\dfrac{(4\omega_{01}^2+3\omega_{01}'^2)}{144V_2}$,

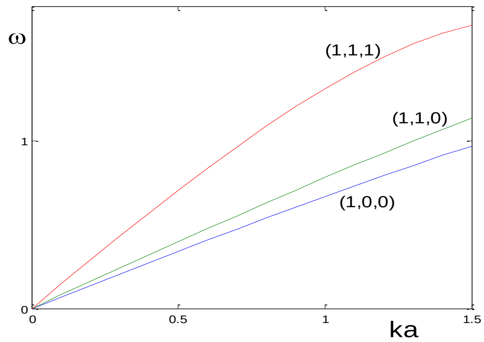
(1,1,1): $P_3=\dfrac{(24\omega_{01}^2+81\omega_{01}'^2+243a\sqrt{3}\omega_{02}^2+27a\omega_{02}'^2)}{36a^2V_3}$, $Q_3=\dfrac{(61\omega_{01}^2+12\omega_{01}'^2)}{144V_3}$.

On introducing new variable $\varsigma=\xi-u_0\tau$, where u_0 is a constant velocity, the soliton solution of KdV equation in the stationary frame is given as

$$U(\xi,\tau) = (3u_0 / P)\mathrm{sec}h^2\left[(\xi - u_0\tau)/2\sqrt{Q/u_0}\right]. \quad (15)$$

The quantities $3u_0/P$ and $2\sqrt{Q/u_0}$ representing peak amplitude and width of the soliton depend on physical situation of system.

Figure 1(a,b,c) shows the longitudinal dispersion relations (14)-(16) in special directions, i.e. the characteristic vector of the lattice cell (1,0,0), $(1,1,0)/\sqrt{2}$, and $(1,1,1)/\sqrt{3}$. Solid lines indicate dispersion relations up to the first neighbor interactions, and dashed lines show dispersion relations up to the second neighbor interactions. It is obvious that dispersion relations and velocities (slop of dispersion curve indicates the velocity of wave) will change when the second neighbor interactions come to our calculations.



The normalized frequency of the longitudinal dust-lattice wave as a function of the normalized wave-number, for wave propagating along (1,0,0), (1,1,0), (1,1,1) directions.

REFERENCES

[1] H. Ikezi, Phys. Fluids **29**, 1764 (1986).

[2] J. H. Chu and Lin I, Phys. Rev. Lett. **72**, 4009 (1994).

[3] S. Nunomura, D. Samsonov, and J. Goree, Phys. Rev. Lett. **84**, 5141 (2000).

[4] X.Y. Wang, W. S. Duan, M. M. Lin, G. X. Wan, Chaos, Solitons and Fractals, **30**, 909 (2006).