# Bounce resonance effect on GAM damping in tokamaks

A.G. Elfimov, R.M.O. Galvão, R.J. F. Sgalla,
*Institute of Physics, University of San Paulo, San Paulo, 05508-090 Brazil*
and A.I. Smolyakov,
*University of Saskatchewan, 116 Science Place, Saskatoon, S7N 5E2 Canada*

Geodesic Acoustic Modes (GAM) are linear eigen-modes (and/or zonal flows) supported by plasma compressibility in toroidal geometry, are linearly coupled to drift-waves via toroidal side-bands of the plasma perturbations, and intrinsically involve anisotropic perturbations of plasma pressure [1-4]. These modes are expected to play an important role in dynamics of drift-wave turbulence. The standard GAMs represent three limit cases of eigenmodes: high frequency geodesic mode (GAM), ion-sound, and the low frequency over damped mode of the neoclassical equilibrium. Our study of GAM continuum is limited to large safety factor ($2q_t^2$ >>1), in large aspect ratio tokamaks $\varepsilon = r/R_0 << 1$, with circular magnetic surfaces $R=R_0(1+\varepsilon\cos\theta)$ where $R_0$ is the major radius. The respective geodesic continuum is usually calculated from the equation $\nabla\cdot\mathbf{j}=0$, which is reduced to the equation $\partial\langle\varepsilon R j_r\rangle_\vartheta/\partial r = 0$ after averaging over magnetic surface. The geodesic oscillations of the current are compensated by the radial polarization current [4] $j_p = -\mathrm{i}\omega c^2 E_1/4\pi c_A^2$, directly produced by the oscillating radial field, where $c_\mathrm{A} = B/\sqrt{4\pi M n_0}$ is the Alfvén velocity. A typical GAM continuum for M=±1, N=0 is given by the equation with drift corrections similar to [1]

$$\omega_\mathrm{geo}^2 \approx \left[\frac{7}{2} + \sum_\pm \chi_\pm \frac{T_e}{T_i}\left(1 \mp \frac{\omega_n^*\mu}{\omega}\right)\right]\frac{v_{Ti}^2}{R_0^2},\ \omega_n^* = \frac{v_{Ti}^2}{r\omega_{ci}n_0}\frac{\partial n_0}{\partial r},\ \mu = \frac{\partial \ln nT}{\partial \ln n},\ \omega_T^* = (\mu-1)\omega_n^* \quad (1)$$

where $$\chi_\pm^{-1} = \left\{1 + \frac{T_e}{T_i}\left[1 \mp \frac{\omega_T^*}{\omega} + \xi_\pm\left(1 \mp \frac{\omega_n^* + (\xi^2 - 1/2)\omega_T^*}{\omega}\right)Z(\xi_\pm)\right]\right\},\ \xi = \frac{R_0 q_t \omega}{\sqrt{2}\mathrm{v}_\mathrm{Ti}},$$

$Z(\xi_{\pm1}) = \int_{-\infty}^{\infty} dt \exp(-t^2)/(t-\xi_{\pm1})$ is the dispersion function, $v_{Te,i} = \sqrt{T_{e,i}/m_{e,i}}$ are the thermal velocities and $T_\mathrm{e,i}$ is electron or ion temperatures, respectively. The χ-function is proportional to dimensionless inversed parallel dielectric tensor component, which is responsible for Landau damping. In cold ion plasmas $T_e>>T_i$, the electron response becomes sufficiently electromagnetic and GAM has that significant components due to electron drift motion $\omega_\pm^2 \approx (7/4 + 2T_e/T_i)v_{Ti}^2/R_0^2 + (\omega_n^*)^2 \pm \sqrt{(2\mu+\mu_0)T_e/T_i}\,|\omega_n^*|\,v_{Ti}/R_0$, $\mu_0 \approx T_e v_{Ti}^2/T_i(R_0\omega_n^*)^2$.

In this work, a method of solution of drift kinetic equation (alternative to [8]) is presented for analyzing of the parallel kinetic response of trapped-untrapped particles for GAM modes. We describe the Landau damping of GAMs using a method as presented in Ref 5-7. The bounce resonance effect of trapped and untrapped particles on the GAM continuum is taken into account via Jacobi functions by solving the drift-kinetic equation as in Ref 5-8

$$\frac{\partial f_{0s}}{\partial\vartheta}-\mathrm{i}\frac{s\omega}{k_0 v_T u}\sqrt{\frac{(1+\varepsilon\cos\vartheta)}{(1+\varepsilon\cos\vartheta-\lambda)}}f_{0s}=\frac{e_{ei}}{T_{ei}k_0}\left[\hat{F}_0E_3-\frac{suv_T}{\omega_c R}\frac{(1+\varepsilon\cos\vartheta-\lambda/2)F_0E_1\sin\vartheta}{\sqrt{1+\varepsilon\cos\vartheta-\lambda}\sqrt{1+\varepsilon\cos\vartheta}}\right] \quad (2)$$

where $\hat{F}_0E_3=F_0E_3-\frac{v_\perp}{\omega_c\omega}\frac{\partial F_0}{r\partial r}\frac{\partial E_3}{\partial\vartheta}$, $v_\perp=\frac{v_{Tei}u\sqrt{\lambda}}{\sqrt{1+\varepsilon\cos\vartheta}}$, $v_\parallel=sv_{Tei}u\sqrt{\frac{(1+\varepsilon\cos\vartheta-\lambda)}{(1+\varepsilon\cos\vartheta)}}$ are velocity space variables, s=±1 and the perturbed distribution for GAMs is $\propto\exp(\pm\mathrm{i}\vartheta-\mathrm{i}\omega t)$.

The χ-term of GAM dispersion in eq.(1) is produced by electron oscillations. In the adiabatic approach, the electron distribution function has the form $f_0^e=F_0^e\,\tilde{n}_i/n_i$ where we assume maxwellian equilibrium distributions for electrons and ions. To calculate the GAM frequency, we use the equation for the averaged radial current

$$\left\langle\tilde{j}_r^{e,i}\right\rangle_\vartheta=(e/4)v_{Te}^4\sum_{s=\pm1}\int_0^\infty u^3du\oint(1+\varepsilon\cos\vartheta)\left(\int_0^{1-\varepsilon}+\int_{1-\varepsilon}^{1+\varepsilon\cos\vartheta}\right)\sqrt{\lambda}\Big/\sqrt{1+\varepsilon\cos\vartheta-\lambda}\;f_1^{e,i}d\lambda d\vartheta \qquad (3)$$

where the integration interval is divided in untrapped and trapped particles regions at $\lambda=1-\varepsilon$. Using first order Larmor radius corrections [9] for the perturbed distribution function $f=f_0+f_1\cos\sigma+f_2\sin\sigma$ proportional to $\exp(\pm\mathrm{i}\vartheta-\mathrm{i}\omega t)$, where $\sigma$ is the angle in the velocity space, the radial distribution function [9] for electrons is presented the form $f_1^e=-\mathrm{v}_\perp(1+\varepsilon\cos\vartheta)f_0^e\cos\vartheta/2\omega_{c0}r$. Then, integrating eq.(3) for electrons we obtain $\tilde{j}_r^e=ev_{Te}^2\tilde{n}_i/\omega_{c0}R_0$, which agrees with previous results [1-4] where cylindrical velocity space coordinates are used. To obtain bounce particles effect in $\tilde{n}_i$, we use a method of solution of eq.(2) via Jacobi functions [5-7,10]. Changing variables $\lambda=1+\varepsilon-2\varepsilon/\kappa^2$ for untrapped (u) particles, $\lambda=\lambda=1+\varepsilon-\hat{\kappa}^2/2\varepsilon$ for the trapped (t) ones and introducing the new variable

$w(\eta)=\int_0^\eta\frac{d\vartheta/2}{\sqrt{1-(\kappa\sin\vartheta/2)^2}}$ or $\eta=2\mathrm{am}(w)+\pi\frac{w}{K}$, we get the untrapped particle solution

$$f_{s,p}^{\mathrm{u},e,i}=-\frac{\mathrm{i}e_{ei}}{T_{ei}}\frac{4K^2F_0}{\pi^2k_0}\sum_{l,m=\pm1}\left[\frac{uA_l^mA_l^pE_3^m}{(lu-s\kappa\Omega_u)}-su^2v_T\frac{(\kappa^2-\varepsilon\kappa^2+2\varepsilon)}{2\sqrt{2\varepsilon}\omega_c\kappa R}\frac{A_l^mB_l^pE_1}{(lu-s\kappa\Omega_u)}\right] \qquad (4)$$

where $\Omega_u = \dfrac{\omega}{k_0 v_T \sqrt{2\varepsilon}}$, $k_0 = \dfrac{B_\vartheta}{rB_0}$, $A_l^m = \dfrac{1}{2K}\int_0^{2K} \mathrm{dn}(w)\exp\left[2im\mathrm{am}(w) + (m-l)\dfrac{\pi w}{K}\right]dw$,

$B_l^m = \int_0^{4K}\dfrac{dw}{4K}\exp\left[2im\cdot\mathrm{am}(w) + (m-l)\dfrac{\pi w}{K}\right]$. Now, using q-series of am/dn-functions and ignoring side band harmonics in $A_l^m$-coefficients, we get $K\approx\pi/2$ for the elliptic integral of first kind and $A_l^m \approx B_l^m = J_{l-m}(\beta_m)$ $\beta_m = 4|m|q$, m=±1 where the small parameter in expansion is $q = \dfrac{1-\sqrt[4]{1-\kappa^2}}{2\left(1+\sqrt[4]{1-\kappa^2}\right)} \approx \dfrac{\kappa^2}{16}\left(1+\dfrac{\kappa^2}{2}\right)$. Then, we use the adiabatic approach $eE_3 = T_e\hat{k}_\parallel\tilde{n}_i/n_i$ for electrons to calculate the ion density oscillations,

$$\tilde{n}_i^{\mathrm{u}} = \frac{i\sqrt{2}e_i n_0 \chi E_1}{2v_{Ti}Mk_0\omega_{ci}R}\left\{(1+2\xi^2)Z(\xi) - \xi\left(\frac{(1+2\varepsilon)}{\sqrt{2\varepsilon}} - 2\right) - \left[\xi^2\left(\frac{1}{2\varepsilon}+1\right)+1\right]Z\left(\frac{\xi}{\sqrt{2\varepsilon}}\right)\right\} \qquad (5)$$

We note that there is only the small correction $n_i^{\mathrm{u}} \approx n_0(1-3\sqrt{2\varepsilon}/4)$ of the untrapped density oscillation to the one calculated in cylindrical velocity space coordinates in the limit $\xi$>>1. For trapped particles, changing the angle variables $\sin\eta = (1/\hat{\kappa})\sin(\vartheta/2)$ and using the new variable $w(\eta) = \int_0^\eta \dfrac{d\eta'}{\sqrt{1-(\hat{\kappa}\sin\eta')^2}}$ in eq. (3), we get the general solution for electrons or ions

$$f_{s,l}^{(\mathrm{t},ei)} = -2i\frac{e}{T}\frac{KF_0}{\pi k_0}\sum_{m,p,s=\pm1}\left[2\frac{u\hat{\kappa}C_l^m E_3^m}{(lu-s\Omega_t)} - su^2 v_T\frac{(1-\varepsilon+2\varepsilon\hat{\kappa}^2)}{\sqrt{2\varepsilon}\omega_c R_0}\frac{mD_l^m E_1}{(lu-s\Omega_t)}\right]\exp\left(il\frac{w}{2K}\right) \qquad (6)$$

$$C_l^m = \int_0^{4K}\frac{dw}{4K}\mathrm{cn}(w)\exp\left[2im\hat{\kappa}\int_0^w \mathrm{cn}dw' + (m-l)\frac{\pi w}{2K}\right], D_l^m = \int_0^{4K}\frac{dw}{4K}\exp\left[2im\mathrm{cn}(w) + (m-l)\frac{\pi w}{2K}\right].$$

Using Jacobi function properties in deeply trapped approach $\hat{q} << 1$, we obtain $K \approx \pi/2$, $C_l^m = \left(4lJ_l(\beta_m^t)\sqrt{\hat{q}}/\hat{\kappa}\beta_m^t\right)$, $D_l^m = J_l(\beta_m^t)$, $\beta_m^t \approx 8\sqrt{\hat{q}}m$, $\Omega_t = \omega/\omega_t$ $\omega_t = \sqrt{\varepsilon/2}k_0 v_{Ti}$ is bounce frequency. Integration of eq. (6) for oscillation density gives for trapped ions

$\tilde{n}_i^{\mathrm{t}} = \dfrac{i\sqrt{2\varepsilon}e_i n_i \chi E_1}{2v_{Ti}Mk_0\omega_{ci}R}\left\{\left(\dfrac{3}{2} + \dfrac{\omega^2}{\omega_t^2} + \dfrac{1}{\sqrt{2}}\left(\dfrac{\omega}{\omega_t}\right)^3 Z\left(\dfrac{\omega}{\sqrt{2}\omega_t}\right)\right)\right\}$ and combination with untrapped ones

$$\chi^{-1} = \left\{1 + \frac{T_e}{T_i}\left[1 + \xi Z(\xi) - \sqrt{2\varepsilon} - \xi Z\left(\frac{\xi}{\sqrt{2\varepsilon}}\right) + \frac{\sqrt{2\varepsilon}}{2}\left(1 + \frac{\omega^2}{\omega_t^2} + \frac{1}{\sqrt{2}}\left(\frac{\omega}{\omega_t}\right)^3 Z\left(\frac{\omega}{\sqrt{2}\omega_t}\right)\right)\right]\right\} \qquad (7)$$

gives the $\chi$-function, which is similar to results [8]. In cold ion limit $\omega^2/2k_0^2 v_{Ti}^2 >> 1$, integration without drift corrections may be performed directly without division in trapped and untrapped regions the distribution function in eq.(3) in the form $\tilde{f}_0^i = -\mathrm{i}\frac{e_i F_0^i}{2M\omega}\frac{u^2\lambda}{\omega_{ci}R}\frac{(2+2\varepsilon\cos\vartheta-\lambda)E_1\sin\vartheta}{(1+\varepsilon\cos\vartheta)}$ and $\tilde{n}_i = -\mathrm{i}\frac{2e_i n_0 E_1 \sin\vartheta}{M\omega_{ci}R_0\omega}$ that gives result in eq. (1). The effect of bounce particle part on GAM dissipation is found to be very small due to the terms $\propto \exp(-\xi^2/\varepsilon)$. In the hot limit $\xi^2/2\varepsilon << 1$, the density oscillations $\tilde{n}_i^u = \mathrm{i}\frac{e_i n_i E_1 \omega \sin\vartheta}{M\omega_{ci}R_0 k_0^2 c_S^2\sqrt{2\varepsilon}}$ are reduced due to this parameter, the oscillation changes phase and the geodesic mode may disappear. The ion radial distribution function adopted for continuum calculations has view $f_1^i = \frac{\sqrt{\lambda}uv_T\sqrt{1+\varepsilon\cos\vartheta}}{\varepsilon\omega_c R}\left[\varepsilon\left(2-\frac{\lambda}{(1+\varepsilon\cos\vartheta)}\right)\frac{\partial f_0^i}{\partial\lambda}\sin\vartheta - f_0^i\cos\vartheta\right]$

General kinetic calculations are difficult to carry out for the ion non-adiabatic part of the geodesic field shown in eq.(5) and integration of the radial current by parts produces a term proportional to $1/(1+\varepsilon\cos\vartheta-\lambda)^{3/2}$ , which diverge for trapped particles. To avoid the problem, we assume the well-known deeply trapped particle approach together with well circulating untrapped particles. In this case, we suggest that the perturbed distribution function is smooth in the transition region between trapped and untrapped particles during integration of the radial current. Due to that the main effect in integration is produced by untrapped particles at $\lambda << 1$ and at $\lambda \le 1+\varepsilon\cos\vartheta$ by the trapped ones.

Finally, we conclude that

- Strong plasma pressure gradients may modify ion-sound branch of geodesic continuum that may stimulate the drift instability.
- Using Jacobi function, the drift-kinetic equation is successfully solved and dissipation of the geodesic modes in trapped-untrapped particles is obtained in analytical form.
- Trapped-untrapped bounce resonances have weak effect on GAMs in plasmas with cold ion, but high temperature impurities may diminish GAM frequency.

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