Theory and Modeling of Planetary Dynamos

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The dipole moment in homogeneous MHD turbulence is $\boldsymbol{\mu}(\mathbf{x}_o) \equiv \int(\mathbf{x}-\mathbf{x}_o)\times\mathbf{j}(\mathbf{x}-\mathbf{x}_o)d^3x$, where $\mathbf{j}(\mathbf{x})$ is electric current and $\mathbf{x}_o$ is a coordinate origin. The mean over all $\mathbf{x}_o$ in a $(2\pi)^3$ periodic box is $[\boldsymbol{\mu}] = \int\boldsymbol{\mu}(\mathbf{x}_o)d^3x_o = 0$, while the same averaging for $\mu^2 = \mu_x^2 + \mu_y^2 + \mu_z^2$ produces

$$[\mu_x^2] \cong \frac{1}{2}[E_M(\hat{\mathbf{y}}) + E_M(\hat{\mathbf{z}})], \quad [\mu_y^2] \cong \frac{1}{2}[E_M(\hat{\mathbf{z}}) + E_M(\hat{\mathbf{x}})], \quad [\mu_z^2] \cong \frac{1}{2}[E_M(\hat{\mathbf{x}}) + E_M(\hat{\mathbf{y}})], \qquad (1a,b,c)$$

$$E_M(\hat{\mathbf{x}}) = |b_y^R(\hat{\mathbf{x}})|^2 + |b_z^R(\hat{\mathbf{x}})|^2, \quad E_M(\hat{\mathbf{y}}) = |b_z^R(\hat{\mathbf{y}})|^2 + |b_x^R(\hat{\mathbf{y}})|^2, \quad E_M(\hat{\mathbf{z}}) = |b_x^R(\hat{\mathbf{z}})|^2 + |b_y^R(\hat{\mathbf{z}})|^2. \qquad (1e,f,g)$$

Above, $\mathbf{k} = \hat{\mathbf{x}}, \hat{\mathbf{y}}$ or $\hat{\mathbf{z}}$, i.e., $k = 1$, and $E_M(\hat{\mathbf{x}}), E_M(\hat{\mathbf{y}}), E_M(\hat{\mathbf{z}})$ are modal magnetic energies.

In this homogeneous case, the $[\mu_i^2]$, $i = x, y, z$, in (1a-c) are expressed in terms of the magnetic energies for only the $k = 1$ modes because these are so large compared to $k \neq 1$ modes [1]. An example of the time evolution of these modal energies is given for the 32^3 Fourier method ideal MHD simulation R32-1 in Fig. 1. Run R32-1 ran on a single processor CPU for 4×10^6 time-steps (with $\Delta t = 0.001$), had a constant energy of $E = 1$, a constant magnetic helicty of $H_M = 0.524$, and a rotation rate of $\Omega_o = 1.0$ about the z-axis. (Runs typically have millions of time-steps in order that the associated dynamical system reach equilibrium and time-averages are meaningful when compared to ensemble averages drawn from statistical theory [1].) In Fig. 1a, over the last half of the run, the time-averages, $\overline{E}_M$, $\overline{H}_C$, plus or minus the standard deviations, are $\overline{E}_M \approx 0.7619 \pm 0.0014$ and $\overline{H}_C \approx (2.02 \pm 1.58)\times10^{-3}$.

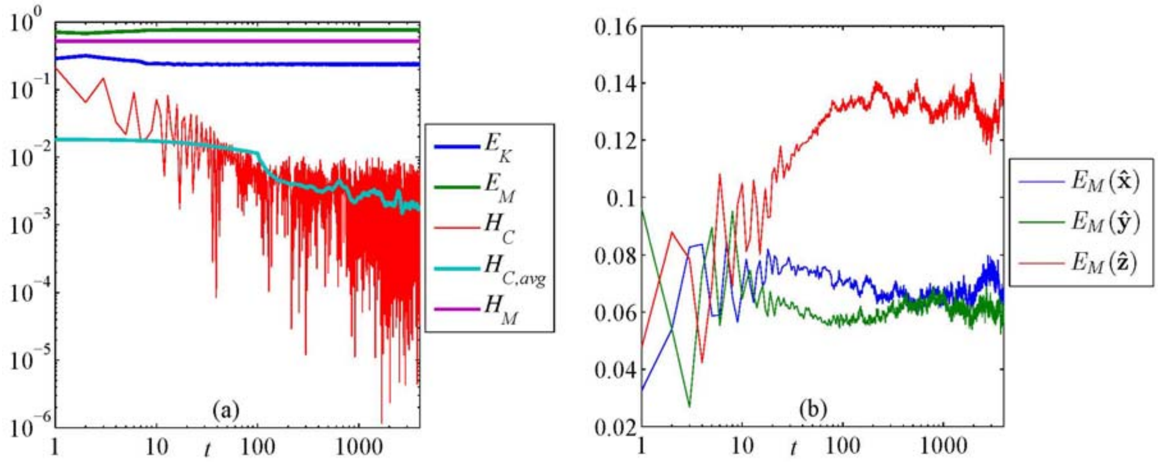

Figure 1. Data from Run R32-1, where $E = 1$, $H_M = 0.5239$, $\boldsymbol{\Omega}_o = 1.0\,\hat{\mathbf{z}}$.

The theoretical prediction is $\langle E_M \rangle = 0.7622$, matching numerical data extremely well. Initially, it was assumed [2] that $\overline{H}_C \approx 0$ for rotating MHD turbulence, but we found in Run R32-1 and several other runs with $0.1 \le H_M \le 0.7$ that we typically have $\overline{H}_C \approx 0.002$, which also occurs in 64^3 simulations; the standard deviation is also about 0.002 in all cases. In Fig. 1b, $E_M(\hat{\mathbf{z}})$ is around twice the size of $E_M(\hat{\mathbf{x}})$ or $E_M(\hat{\mathbf{y}})$ at equilibrium, and the energy in the $k = 1$ modes of the magnetic field is about 28% of the total energy E. To see what happened in the presence of dissipation, we began with the same initial conditions as ideal run R32-1 and ran a simulation with $\nu_o = \eta_o = 0.001$, which we named R32-2, for $5{\times}10^5$ Δt's, again with $\Delta t = 0.001$. Thus, while R32-1 ran from simulation time t = 0 to 4000, R32-2 ran from t = 0 to 500, at which point dissipation had moved the system out of its turbulent stage and into its essentially purely decaying stage, with E falling from $E = 1$ at $t = 0$, to $E = 0.1875$ at $t = 500$.

Although the total energy $E(t)$ decayed in R32-2, the quantities $E_M(\hat{\mathbf{x}},t)/E(t)$, $E_M(\hat{\mathbf{y}},t)/E(t)$ and $E_M(\hat{\mathbf{z}},t)/E(t)$ behaved similarly to what appears in Fig. 1b. However, $|H_M(t)| \to E_M(t)$ and $E_M(t)/E(t) \to 1$ as t increased in R32-2, so that E_M and H_M decayed more slowly than $E(t)$, i.e., 'selective decay' [3] occurred. Thus, in R32-2, and other dissipative, homogeneous MHD turbulence simulations, we observe that energy tends to become essentially all magnetic and to reside in the $k = 1$ modes, i.e., energy concentrates at the longest wavelengths and H_M/E tends to maximize. In R32-2, $E_K(t)$ falls to less than 1% of $E_M(t)$ at about $t = 160$, while in the ideal R32-1, at equilibrium, $E_M(t) = 0.7622$ and $E_K(t) = 0.2378$ at $t = 4000$. In contrast, in simulations without rotation, $E_K(t)/E_M(t) \sim 2/3$ for both ideal and decaying runs. Runs may be compared by plotting $\mathrm{Re}\,\tilde{b}_n(\mathbf{k},t)$ vs $\mathrm{Im}\,\tilde{b}_n(\mathbf{k},t)$, $n = x, y, z$, as is shown in Fig. 2. It is clear that ideal and decaying $\tilde{\mathbf{b}}(\mathbf{k},t)$ with $k = 1$ are very similar until late in the run, when dissipation causes the R32-2 components to enter a linear decay stage. Fig. 2 also demonstrates 'broken ergodicity' [1,2], since ensemble predictions are that all $\langle \tilde{\mathbf{b}}(\mathbf{k},t) \rangle = 0$.

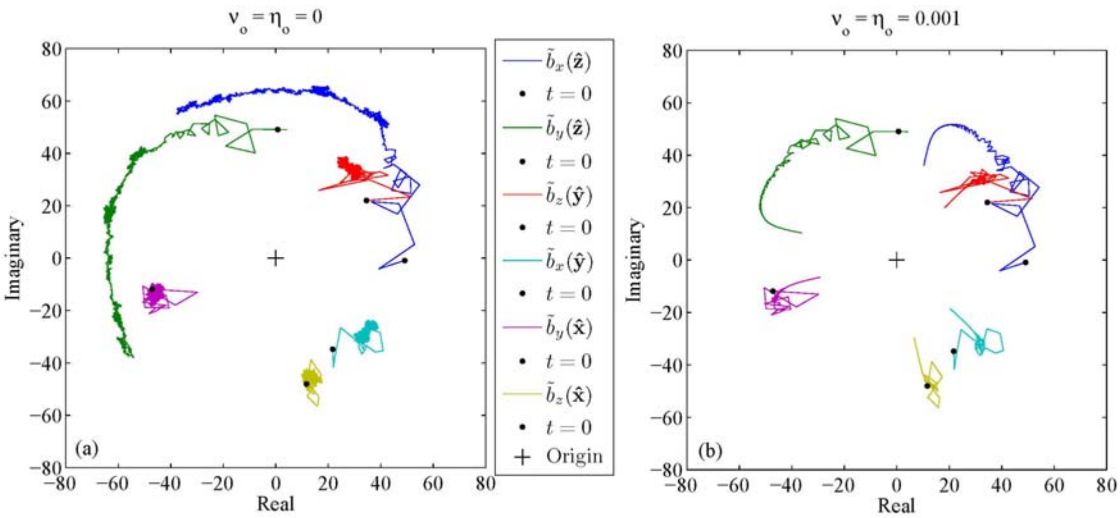

Figure 2. Phase plots of $\tilde{\mathbf{b}}(\mathbf{k})$ with $k = 1$ for (a) ideal run R32-1 and (b) dissipative run R32-2.

Hybrid Statistics

In order to explain some apparent differences between the statistical theory of 2-D ideal MHD turbulence in the presence of a mean magnetic field $\mathbf{B}_o$, the concept of hybrid statistics was introduced [4]. Here, we apply hybrid statistics to the case of 3-D MHD turbulence in a frame of reference rotating with angular velocity $\mathbf{\Omega}_o = \Omega_o \hat{\mathbf{z}}$. The periodic box, Fourier method results shown in Figure 1b suggest that the statistical theory of ideal, rotating 3-D MHD turbulence [1,2] must be modified so that ensemble predictions for modes with $\mathbf{k} = (0, 0, k_z)$, which are symmetric upon rotation about the z-axis, are qualitatively different from those with $\mathbf{k} = (k_x, k_y, k_z)$, where at least one of either k_x or k_y is nonzero, so that the associated modes are not symmetric upon rotation about the z-axis. In order to incorporate anisotropy in the statistical theory, we must make use of the available parameters, i.e., H_M and H_C in the ideal case, and H_M/E and H_C/E in the decaying case, treated as a quasiequilibrium. As Fig. 2 shows, this is not an unreasonable assumption.

Here, we will consider directly the development of hybrid statistics for ideal MHD turbulence in a rotating spherical shell, which is modeled using spherical Bessel-spherical harmonic expansions [5]. In the spherical shell model system, the $m = 0$ modes of the expansions are rotationally symmetric about the z-axis, while the $m \neq 0$ modes are not. The three $l = n = 1$, modes with $m = 0, \pm 1$ (i.e., dipole components) are the most energetic and the most affected by hybrid statistics, i.e., exhibit the strongest anisotropy. The principle of hybrid statistics is applied as follows to rotating, ideal MHD turbulence in a spherical shell. The quantities $\hat{a}$, $\hat{\beta}$, and $\hat{\gamma}$ that appear in modal ensemble predictions have a general form that depends on the two parameters $\hat{\mathcal{H}}_C$ and $\hat{\mathcal{H}}_M$, along with the undetermined variable $\hat{\phi}$, which is found by minimizing an 'entropy functional' [5] with respect to $\hat{\phi}$, a minimum that occurs at $\hat{\phi} = \hat{\phi}_o$. Anisotropy is introduced by assigning all $m = 0$ modes an effective value $\hat{\mathcal{H}}_C \approx 0.002$ for ideal MHD, and an effective value $\hat{\mathcal{H}}_C \approx 0.02$ for decaying MHD; these approximate values appear to emerge consistently from 32^3 and 64^3 periodic box simulations. In turn, all $m \neq 0$ modes are nominally assigned an effective value $\hat{\mathcal{H}}_C = 0$. Two sets of $\hat{a}$, $\hat{\beta}$, and $\hat{\gamma}$, one for all $m = 0$ modes ($\hat{a}_0$, $\hat{\beta}_0$, and $\hat{\gamma}_0$) and the other for all $m \neq 0$ modes ($\hat{a}_m$, $\hat{\beta}_m$, and $\hat{\gamma}_m$) are needed:

$$\hat{\alpha}_0 = \frac{\hat{\phi}_o}{\hat{\phi}_o\left(1-\hat{\phi}_o\right) - \hat{\mathcal{H}}_C^2}, \qquad \hat{\beta}_0 = -2\frac{\hat{\mathcal{H}}_C}{\hat{\phi}_o}\hat{\alpha}_0, \qquad \hat{\gamma}_0 = -\frac{2\hat{\phi}_o - 1}{\hat{\mathcal{H}}_M}\hat{\alpha}_0, \qquad m = 0, \quad \text{(2a,b,c)}$$

$$\hat{\alpha}_m = \frac{1}{1-\hat{\phi}_o}, \qquad \hat{\beta}_m = 0, \qquad \hat{\gamma}_m = -\frac{2\hat{\phi}_o - 1}{\hat{\mathcal{H}}_M\left(1-\hat{\phi}_o\right)}, \qquad m \neq 0. \quad \text{(2c,d,e)}$$

Here, $\hat{\mathcal{H}}_C = \mathcal{H}_C/\mathcal{E}$, $\hat{\mathcal{H}}_M = \mathcal{H}_M/\mathcal{E}$, where $\mathcal{E}$, $\mathcal{H}_C$, and $\mathcal{H}_M$, represent specific values of energy, cross and magnetic helicity, and $2M$ is the number of independent coefficients a simulation. The 'undetermined value' of the normalized magnetic energy is $\hat{\phi} = \mathcal{E}_M/\mathcal{E}$ and the expected value $\hat{\phi} = \hat{\phi}_o$ is found by a minimization procedure and fixes the $\hat{a}_m$, $\hat{\beta}_m$, and $\hat{\gamma}_m$ in (2a-e), allowing for ensemble predictions of the modal energies $\langle|b_{lmn}|^2\rangle$. The magnetic energies for those modes with $l = n = 1$ are expected to be ~M times greater than for any other modes [5], i.e., the magnetic energy spectrum peaks sharply at l =1. The ratio $B \equiv \langle|b_{101}|^2\rangle/\langle|b_{111}|^2\rangle$ between the $m = 0$ and 1 modal magnetic energies for $l = n = 1$ can now also be predicted.

Using results in [5], we can find the ratio B, and from it, the dipole angle $\theta_D \equiv \arctan(2/B^{1/2})$; we plot this for several values of $\hat{\mathcal{H}}_M$ over the range $10^{-5} \le \hat{\mathcal{H}}_C \le 0.5$ in Fig. 3.

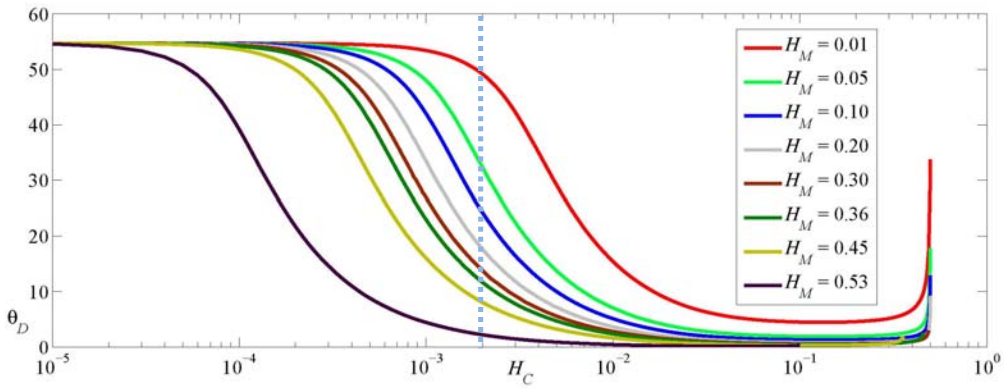

Figure 3. Expectation values for dipole angle θ_D in a model ideal MHD turbulent geodynamo.

The results shown in Fig. 3 come from a theoretical model of ideal (E = 1) MHD turbulence in a rotating spherical shell using an expansion with $1 \le l \le 100$, $-l \le m \le l$ and $1 \le n \le 100$. The vertical blue dotted line indicates $H_C \approx 0.002$ and the maximum value possible for H_M is found from the relations $k_{11}|H_M| \le E_M \le E$, with k_{11} = 1.8638, if an Earth-like ratio of outer and inner radii for the liquid core are used. Since E = 1, maximum $|H_M|$ = 0.53654; however, in the case of periodic box MHD turbulence, the lowest spherical wavenumber k_{11} is replaced by the lowest Fourier wavenumber k = 1, in which case we have maximum $|H_M|$ = 1. Please remember that the results shown in Fig. 3 are statistical averages for ideal systems in equilibrium and decaying systems in quasiequilibrium at the larger scales. Understanding dynamical phenomena, such as excursions and polarity reversals in a geodynamo, requires numerical simulations and, if possible, a dynamical theory in addition to a statistical one.

References

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