On the structure of Maxwell's equations in the region of linear coupling of electromagnetic waves in weakly ingomogeneous plasmas

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1. Introduction

Linear coupling of electromagnetic waves in weakly inhomogeneous media is a very important fundamental process in plasma physics, crystal optics, electrodynamics of metamaterials, etc. In each of these fields, the linear coupling is usually considered taking into account the specific properties of the dielectric response and particular geometry. In this paper, we aim to look at the problem at a new angle, we consider a linear medium with a dielectric permittivity tensor of the general form in order to clarify the conditions that the dielectric response must satisfy to ensure the existence of effectively interacting modes in an unbounded weakly inhomogeneous medium. We proceed from the assumption that linear coupling between two electromagnetic waves can occur only in the vicinity of polarization degeneracy points with two linearly independent solutions of the Maxwell equations for a single wave vector $\mathbf{k}(\omega)$. This condition, which can actually be regarded as a definition of the linear coupling, is in itself sufficient to impose strong constraints on the dielectric tensor components and offer a universal classification of the possible types of linear wave coupling independent of a concrete medium model. Moreover, such an approach permits overcoming a number of difficulties encountered in the theory of linear coupling of electromagnetic waves in the case of multidimensional strongly anisotropic and gyrotropic media.

Modern mode-coupling theories extensively use the approximation of coupled waves,

$$\hat{D}_1 E_1 = \eta E_2 ,\ \hat{D}_2 E_2 = \eta^* E_1 , \tag{1}$$

where $\hat{D}_1$ and $\hat{D}_2$ are operators describing the propagation of geometric-optical modes with complex amplitudes E_1,E_2 in a weakly inhomogeneous medium whose properties only slightly change over distances of the order of the wavelength, and η is a mode-coupling constant which is non-zero inside the interaction region [1]. However the Maxwell equations in weakly inhomogeneous anisotropic and gyrotropic media do not always lead to the above mentioned equations for interacting waves because $\eta \to 0$ for a sufficiently wide range of problems. In this case, the coupling constant must be substituted by a differential operator that accounts for the essentially non-one-dimensional character of the linear wave coupling in free space [2-4].

The method described in this paper allows thoroughly studying the conditions of applicability of the widely used approximation of coupled waves in weakly inhomogeneous media and demonstrating the limitations of this model in many cases important for applications.

2. Dielectric permittivity tensor at the polarization degeneracy point

Let us consider a homogeneous medium specified by dielectric tensor $\varepsilon_{ij}(\omega)$ in a Cartesian coordinate system. For a plane electromagnetic wave propagating in this medium, $\tilde{\mathbf{E}} = \mathbf{E}\exp(i\mathbf{kr} - i\omega t)$, the Maxwell equations and corresponding dispersion equation are

$$\left(k^2\delta_{ij} - k_i k_j - k_0^2\varepsilon_{ij}\right)E_j = 0\,, \quad \det\left(k^2\delta_{ij} - k_i k_j - k_0^2\varepsilon_{ij}\right) = 0 \tag{2}$$

where $k_0 = \omega / c$ is the vacuum wave vector, $k = |\,\mathbf{k}\,|$, δ_{ij} is the Kronecker symbol, indices *i*, *j* denote projections onto the Cartesian coordinate axes, and the repeated indices imply summation. We study polarization degeneracy, i.e., the existence of two linearly independent solutions $\mathbf{E}$ of the system of wave equations for certain $\mathbf{k}(\omega)$ satisfying the dispersion relation. This analysis may most easily performed by choosing a new basis for the electromagnetic field, $\mathbf{E} = \mathcal{E}_j\mathbf{e}_j$, where $\mathbf{e}_j = (U_{1j}, U_{2j}, U_{3j})$ is a unitary basis of the *medium eigenpolarizations* in which the dielectric tensor is diagonal, $U_{mi}^{-1}\varepsilon_{ij}U_{jn} = \mathrm{diag}(\varepsilon_1, \varepsilon_2, \varepsilon_3)$. Such representation exists in lossless media, however the absence of dissipation is a sufficient but not necessary condition for the following analysis. Indeed, it remains valid for dissipative medium once its dielectric tensor can be diagonalized, what is possible for example in magnetized plasma with collisions. The triplet $(\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3)$ can formally be regarded as components of a vector $\boldsymbol{\mathcal{E}}$. In the case of polarization degeneracy, there are two linearly independent vectors $\boldsymbol{\mathcal{E}}_a$ and $\boldsymbol{\mathcal{E}}_b$ satisfying (2). Evidently, any linear combination of $\boldsymbol{\mathcal{E}}_a$ and $\boldsymbol{\mathcal{E}}_b$ is also a solution of the set of wave equations. Hence, there is a distinguished direction in the case of polarization degeneracy, $\boldsymbol{\tau} = \boldsymbol{\mathcal{E}}_a \times \boldsymbol{\mathcal{E}}_b$, such that any vector $\boldsymbol{\mathcal{E}} \perp \boldsymbol{\tau}$ is a solution of the wave equation (2). From the standpoint of wave equations, the three eigenvectors of the medium are equivalent, which permits fixing one vector projection, e.g., $\tau_3 = -1$. It then follows from the orthogonality that $\mathcal{E}_3 = \tau_1\mathcal{E}_1 + \tau_2\mathcal{E}_2$, substituting this in (2), one obtains $(D_{i1} + \tau_1 D_{i3})\,\mathcal{E}_1 + (D_{i2} + \tau_2 D_{i3})\,\mathcal{E}_2 = 0$ ($i = 1,2,3$). The latter relations must be satisfied for any $\mathcal{E}_1$ and $\mathcal{E}_2$ what is possible if and only if all coefficients of $\mathcal{E}_1$ and $\mathcal{E}_2$ vanish. Fortunately it is possible to list all possible situations consistent with these conditions. Up to permutations of indices, there are different five cases:

(a) Polarization degeneracy under conditions of partial anisotropic degeneracy,

$$\mathbf{e}_i \parallel \mathbf{k}, \ \ \varepsilon_i \neq 0, \ \ \varepsilon_j = \varepsilon_k = n^2 \ \ (j \neq k \neq i),$$

Polarizations of degenerate modes here are orthogonal to the eigenpolarization vector $\mathbf{e}_i$.

(b) Polarization degeneracy in the vicinity of a medium resonance,

$$\mathbf{e}_i \parallel \mathbf{k}, \ \ \varepsilon_i = 0, \ \ \varepsilon_j = n^2, \ \ \varepsilon_k \neq n^2 \quad (j \neq k \neq i).$$

Polarizations of degenerate modes are orthogonal to the eigenpolarization vector $\mathbf{e}_k$.

(c) Complete polarization degeneracy,

$$\mathbf{e}_i \parallel \mathbf{k}, \ \ \varepsilon_i = 0, \ \ \varepsilon_j = \varepsilon_k = n^2 \ \ (j \neq k \neq i).$$

In this case any field polarization satisfies the system of wave equations.

(d) Complete degeneracy of medium anisotropy, $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = n^2$. Obviously, in absence of spatial dispersion this case may be realized for all directions of wave propagation.

(e) Wave vector orthogonal to the medium eigenpolarization vector,

$$\mathbf{e}_i \perp \mathbf{k}, \ \ \varepsilon_i = \varepsilon_j \varepsilon_k k^2 / [\varepsilon_j |\mathbf{k}\mathbf{e}_j|^2 + \varepsilon_k |\mathbf{k}\mathbf{e}_k|^2] = n^2 \quad (j \neq k \neq i).$$

These cases cover all typical cases of linear interaction in unbounded weakly inhomogeneous media. For example, magnetized plasma has a gyrotropy axis along the magnetic field and is isotropic in the plane across to it; therefore one of the eigenpolarization vectors is real and is directed along the magnetic field, while the other two eigenpolarization vectors are complex. Then it follows from the above conditions that the coupling between waves with real **k** is possible only when the wave vector is parallel to the gyrotropy axis. If the medium is a uniaxial or biaxial crystal, all its three eigenpolarization vectors are real and are directed along the principal optical axes of the crystal. The linear interaction of propagating waves is possible if the wave vector is either parallel or orthogonal to one of the principal optical axes. Finally, if an anisotropic crystal has a gyrotropy direction that does not coincide with the principal optical axes (e.g., the one induced by an arbitrarily directed external magnetic field), then all three eigenpolarization vectors are complex. Then the only possibility for propagating waves to interact linearly is realized as in case (5); the description of this rather exotic case requires a somewhat different formalism [5].

3. Reference wave equations in the vicinity of polarization degeneracy points

Here we consider a weakly inhomogeneous medium varying in space more slowly than the wavelength far from the interaction region, $k_0 L >> 1$ where L is the characteristic scale of variation of dielectric properties. This condition permits distinguishing normal waves propagating independently in the geometric-optics limit, but this approach is broken in the vicinity of the polarization degeneracy points found above, resulting in the linear coupling between normal waves. In this case, the electromagnetic field can be described by reference wave equations derived from the Maxwell equations truncated in the vicinity of polarization degeneracy points (a)-(e). For this, one can again seek the wave field in the eigenpolarization representation, $\mathbf{E}(\mathbf{r},t) = \mathcal{E}_i(\mathbf{r})\,\mathbf{e}_i(\mathbf{r})\exp(i\mathbf{kr} - i\omega t)$, where $\mathcal{E}_i(\mathbf{r})$ are slow functions on the wavelength scale, and $\mathbf{e}_i(\mathbf{r})$ are basis that diagonalize $\varepsilon_{ij}(\mathbf{r})$ at each spatial point. The equations for slow field amplitudes can be derived by substituting in Eqs. (2) a differential operator for the wave vector, $\mathbf{k} \to \hat{\mathbf{k}} = \mathbf{k}^0 - i\partial/\partial\mathbf{r}$, and retaining only the first-order terms over $\partial/\partial\mathbf{r}$ having in mind that in the vicinity of the polarization degeneracy point the spatial derivatives are small compared with the carrier wave vector $\mathbf{k}^0$. In this way we obtain five classes of wave equations that studied in detail in [5]. Here simplest reference forms of such equations are identified, and its application to plasma electrodynamics is discussed.

It is shown that linear coupling of electromagnetic waves in unbounded weakly inhomogeneous media follows two distinct scenarios. In the case of simultaneous anisotropy and gyrotropy degeneracy, the linear wave coupling is realized on a slow scale L determined by the scale of parameter variations [cases (a,d)]. This situation can be interpreted as a scalar coupling of two geometric-optical modes given by model (1). Because the coupling occurs along the rays, the wave conversion is a one-dimensional process. In the vicinity of medium resonances [cases (b,c)] and in the case of transverse propagation [case (e)], the fast wave coupling scenario is realized. The conversion occurs on a new small scale, $L/\sqrt{k_0 L} << L$, at which the geometric-optics approximation is no longer valid. Thus it can not be considered as a coupling of geometric-optical modes and must be described by solutions that may possess essentially non-one-dimensional structure such as asymmetry between forward and backward mode conversion found in [2,4]. Wave coupling in gyrotropic media is typically non-one-dimensional, while in anisotropic media without gyrotropy it can always be described as a one-dimensional (small-scale) process in an effective plane-layered medium [5].

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