Long-wavelength limit of second-order gyrokinetics and the intrinsic ambipolarity of the turbulent tokamak

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Toroidal angular momentum transport in tokamaks has become an extremely active research field. The main cause of radial flux of toroidal angular momentum is microturbulence, and the appropriate framework to describe microturbulence is gyrokinetic theory: a reduced kinetic theory consisting of the elimination of the degree of freedom associated to the gyration of the particle around the magnetic field order by order in an asymptotic expansion in $\varepsilon = \rho/L \ll 1$, where ρ is the typical gyroradius and L is the characteristic length of variation of the background magnetic field. Unfortunately, the toroidal component of the momentum equation that would give the toroidal component of the velocity and completely determine the radial electric field is identically satisfied to order ε^2 by any toroidal velocity [1], whereas gyrokinetic equations are customarily derived and solved to order ε. In Ref. [2], a method to calculate the toroidal angular momentum in the low flow limit to the lowest non-trivial order is proposed. The formula for the radial flux of toroidal angular momentum in the electrostatic case is given as a sum of several integrals over the first and second-order pieces of the distribution functions and the electrostatic potential. The recent derivation of the gyrokinetic equations and change of coordinates to second-order in general, static, magnetic geometry [3] permits us to tackle the problem of formulating a complete model for toroidal momentum transport in turbulent tokamaks.

In this conference contribution we report on the first step towards our goal: the derivation of the equations that need to be solved to obtain the long-wavelength pieces of the fields up to second order. The full computation of such equations allows to give, in parallel, an explicit proof of the indeterminacy of the radial electric field, showing that it cannot be found from the long-wavelength gyrokinetic Fokker-Planck and quasineutrality equations correct to second order, i.e. the turbulent tokamak is intrinsically ambipolar. The following pages are devoted to explain the main points of the argument leading to the proof of the tokamak intrinsic ambipolarity. The details are quite involved and technical and can be found in Ref. [4].

The species-independent normalization

$$\underline{t} = \frac{c_s t}{L},\ \underline{\mathbf{r}} = \frac{\mathbf{r}}{L},\ \underline{\mathbf{A}} = \frac{\mathbf{A}}{B_0 L},\ \underline{\varphi} = \frac{e\varphi}{\varepsilon_s T_{e0}},\ \underline{H_\sigma} = \frac{H_\sigma}{T_{e0}},\ \underline{n_\sigma} = \frac{n_\sigma}{n_{e0}},\ \underline{T_\sigma} = \frac{T_\sigma}{T_{e0}}, \tag{1}$$

is employed for time, space, vector potential, electrostatic potential, Hamiltonian, particle density, and temperature; and the species-dependent normalization

$$\underline{\mathbf{v}_\sigma} = \frac{\mathbf{v}_\sigma}{v_{t\sigma}}, \ \underline{f_\sigma} = \frac{v_{t\sigma}^3}{n_{e0}} f_\sigma, \tag{2}$$

for velocities and distribution functions. In the previous expressions $L \sim |\nabla_{\mathbf{r}} \ln |\mathbf{B}||^{-1}$ is the typical length of variation of the magnetic field, B_0 a typical value of the magnetic field strength, $c_s = \sqrt{T_{e0}/m_i}$ the sound speed, T_{e0} a typical electron temperature, n_{e0} a typical electron density, and m_i the mass of the dominant ion species, that we assume singly charged. Finally, $v_{t\sigma}$ is the thermal speed of species σ, $\varepsilon_s = \rho_s/L$, where $\rho_s = c_s/\Omega_i$ is a characteristic sound gyroradius, and $\Omega_i = eB_0/(m_i c)$ is a characteristic ion gyrofrequency. We take $v_{t\sigma} = \sqrt{T_{e0}/m_\sigma}$ as the expression for the typical thermal speed, i.e. we assume that T_{e0}, the characteristic temperature of electrons, is also the characteristic temperature for all species. The natural, species-independent expansion parameter in gyrokinetic theory is ε_s. Many expressions, however, are more conveniently written in terms of the species-dependent parameter $\varepsilon_\sigma = \rho_\sigma/L$, where $\rho_\sigma = v_{t\sigma}/\Omega_\sigma$ is a characteristic gyroradius of species σ and $\Omega_\sigma = Z_\sigma e B_0/(m_\sigma c)$ a characteristic gyrofrequency. Observe that the relation between ε_σ and ε_s is $\varepsilon_s = \lambda_\sigma \varepsilon_\sigma$, with $\lambda_\sigma = \rho_s/\rho_\sigma = Z_\sigma \sqrt{m_i/m_\sigma}$.

In dimensionless variables, the Fokker-Planck equation reads

$$\partial_{\underline{t}} \underline{f_\sigma} + \tau_\sigma \left\{ \underline{f_\sigma}, \underline{H_\sigma} \right\}_{\underline{\mathbf{X}}} = \tau_\sigma \sum_{\sigma'} \underline{C_{\sigma\sigma'}}[\underline{f_\sigma}, \underline{f_{\sigma'}}](\underline{\mathbf{r}}, \underline{\mathbf{v}}), \tag{3}$$

where $\underline{H}_\sigma = \underline{\mathbf{v}_\sigma}^2/2 + Z\varepsilon_s \underline{\varphi}$, $\tau_\sigma = v_{t\sigma}/c_s = \sqrt{m_i/m_\sigma}$, and the Poisson bracket of two functions $g_1(\underline{\mathbf{r}}, \underline{\mathbf{v}})$, $g_2(\underline{\mathbf{r}}, \underline{\mathbf{v}})$ (we no longer write the subindex σ in $\mathbf{v}_\sigma$) is defined by

$$\{g_1, g_2\}_{\underline{\mathbf{X}}} = \left(\nabla_{\underline{\mathbf{r}}} g_1 \cdot \nabla_{\underline{\mathbf{v}}} g_2 - \nabla_{\underline{\mathbf{v}}} g_1 \cdot \nabla_{\underline{\mathbf{r}}} g_2 \right) + \varepsilon_\sigma^{-1} \underline{\mathbf{B}} \cdot (\nabla_{\underline{\mathbf{v}}} g_1 \times \nabla_{\underline{\mathbf{v}}} g_2). \tag{4}$$

Here $\underline{\mathbf{X}} \equiv (\underline{\mathbf{r}}, \underline{\mathbf{v}})$ are the dimensionless cartesian coordinates. As for the quasineutrality equation:

$$\sum_\sigma Z_\sigma \int \underline{f_\sigma}(\underline{\mathbf{r}}, \underline{\mathbf{v}}, \underline{t}) \mathrm{d}^3 \underline{v} = 0. \tag{5}$$

From now on, we drop the underlining and assume that we work in dimensionless variables.

We want to write the Fokker-Planck and quasineutrality equations in gyrokinetic coordinates $\mathbf{Z} \equiv (\mathbf{R}, u, \mu, \theta)$. The transformation from euclidean to gyrokinetic coordinates, $\mathbf{X} = \mathscr{T}_\sigma(\mathbf{Z}, t)$, up to second order was computed for the first time in [3]. Denote by $\mathscr{T}_\sigma^*$ the pull-back of $\mathscr{T}_\sigma$. Acting on a function $g(\mathbf{X}, t)$, $\mathscr{T}_\sigma^* g(\mathbf{Z}, t)$ is simply the function g written in coordinates $\mathbf{Z}$, i.e. $\mathscr{T}_\sigma^* g(\mathbf{Z}, t) = g(\mathscr{T}_\sigma(\mathbf{Z}, t), t)$. Now, define by $F_\sigma := \mathscr{T}_\sigma^* f_\sigma$ the distribution function in gyrokinetic coordinates. Using the notation $F_\sigma = \sum_{n=0}^\infty \varepsilon_\sigma^n F_{\sigma n}$, one finds [4] that to order zero the distribution function is Maxwellian,

$$F_{\sigma 0}(\mathbf{R},u,\mu,t) = \frac{n_\sigma(\mathbf{R},t)}{(2\pi T_\sigma(\mathbf{R},t))^{3/2}} \exp\left(-\frac{\mu B(\mathbf{R}) + u^2/2}{T_\sigma(\mathbf{R},t)}\right), \tag{6}$$

and that to order one and two the long-wavelength Fokker-Planck equation has the following form:

$$\left(u\hat{\mathbf{b}}\cdot\nabla_{\mathbf{R}} - \mu\hat{\mathbf{b}}\cdot\nabla_{\mathbf{R}} B\partial_u\right) F_{\sigma i}^{\mathrm{lw}} = \sum_{\sigma'} \mathscr{T}_{\sigma,0}^{*} C_{\sigma\sigma'}\left[\mathscr{T}_{\sigma,0}^{-1*} F_{\sigma i}^{\mathrm{lw}}, \mathscr{T}_{\sigma',0}^{-1*} F_{\sigma' 0}\right]$$
$$+\sum_{\sigma'}\left(\frac{\lambda_\sigma}{\lambda_{\sigma'}}\right)^i \mathscr{T}_{\sigma,0}^{*} C_{\sigma\sigma'}\left[\mathscr{T}_{\sigma,0}^{-1*} F_{\sigma 0}, \mathscr{T}_{\sigma',0}^{-1*} F_{\sigma' i}^{\mathrm{lw}}\right] + \dots. \tag{7}$$

Here $F_{\sigma i}^{\mathrm{lw}}$ is the long-wavelength component of $F_{\sigma i}$. We refer the reader to [4] for a thorough discussion on ordering and scale separation issues and for the terms omitted in (7).

If $F_{\sigma i}^{\mathrm{lw}}$, $i = 1,2$ are solutions of the first and second-order Fokker-Planck equations, then so are $F_{\sigma i}^{\mathrm{lw}} + h_{\sigma i}$, $i = 1,2$, where

$$h_{\sigma i} \ = \ \left[\frac{n_{\sigma i}}{n_\sigma} + \left(\frac{\mu B + u^2/2}{T_\sigma} - \frac{3}{2}\right)\frac{T_{\sigma i}}{T_\sigma}\right] F_{\sigma 0}, \tag{8}$$

for an arbitrary set of flux functions $\{n_{\sigma i}(\psi,t), T_{\sigma i}(\psi,t)\}_\sigma$, with the only restriction $T_{\sigma j} = T_{\sigma' j}$, for all σ, σ'. Once we have learnt this, let us turn to the quasineutrality equation. To order ε_s^0 we have $\sum_\sigma Z_\sigma n_\sigma(\mathbf{r},t) = 0$. To order ε_s:

$$\sum_\sigma \frac{Z_\sigma}{\lambda_\sigma} n_{\sigma 1} + \sum_\sigma \frac{Z_\sigma}{\lambda_\sigma} \int B(\mathbf{r}) F_{\sigma 1}^{\mathrm{lw}}(\mathbf{r},u,\mu,t)\,\mathrm{d}u\,\mathrm{d}\mu\,\mathrm{d}\theta = 0, \tag{9}$$

and to order ε_s^2 (again, see [4] for the full expressions):

$$\sum_\sigma \frac{Z_\sigma}{\lambda_\sigma^2} n_{\sigma 2} + \sum_\sigma \frac{Z_\sigma}{\lambda_\sigma^2}\left[\int B F_{\sigma 2}^{\mathrm{lw}}\mathrm{d}u\,\mathrm{d}\mu\,\mathrm{d}\theta + \nabla_{\mathbf{r}}\cdot\left(\frac{Z_\sigma n_\sigma}{B^2}\nabla_{\mathbf{r}}\varphi_0\right)\right] + \dots = 0. \tag{10}$$

Even though $\varphi_0(\psi)$, the lowest-order electrostatic potential, enters the long-wavelength quasineutrality equation to second-order, it cannot be determined. The first and second-order pieces of the long-wavelength quasineutrality equation simply give constraints on the corrections $n_{\sigma 1}$ and $n_{\sigma 2}$. Each function $n_{\sigma i}$ will be determined by a transport equation that appears as a solvability condition for a higher order long-wavelength piece of the Fokker-Planck equation, just as a transport equation for n_σ is derived in [4] and shown below.

The task of proving that the radial electric field is undetermined, or equivalently, that the turbulent tokamak is intrinsically ambipolar, has not been accomplished yet. As in the Chapman-Enskog theory of neutral gases, the equation for the distribution function to order ε_σ^i possesses, in general, solvability conditions. This means that the existence of a solution of the equation to order ε_σ^i implies a new equation involving lower-order quantities (obtained from the solution of the equations to order ε^j, $j < i$). Indeed, the second-order Fokker-Planck equation yields, as solvability conditions, transport equations for the lowest order density and temperature functions n_σ and T_σ. Concretely,

$$\partial_{\varepsilon_s^2 t} n_\sigma(\psi,t) = \frac{1}{V'(\psi)}\partial_\psi \left\langle V'(\psi)\int \mathrm{d}u\mathrm{d}\mu\mathrm{d}\theta \left\{ \left[F_{\sigma 1}^{\mathrm{sw}}(\nabla_{\mathbf{R}_\perp/\varepsilon_\sigma}\langle\phi_{\sigma 1}^{\mathrm{sw}}\rangle \times \hat{\mathbf{b}})\cdot\nabla_{\mathbf{R}}\psi\right]^{\mathrm{lw}} + \frac{B}{Z_\sigma\lambda_\sigma}\left\langle\left(\frac{Iu}{B}+\rho\cdot\nabla_{\mathbf{R}}\psi\right)\sum_{\sigma'} C_{\sigma\sigma'}^{(1)\mathrm{lw}}\right\rangle\right\}\right\rangle_\psi, \tag{11}$$

and

$$\begin{aligned}\partial_{\varepsilon_s^2 t}\left(\sum_\sigma \frac{3}{2} n_\sigma(\psi,t)T_\sigma(\psi,t)\right) = \\ \frac{1}{V'(\psi)}\partial_\psi\left\langle V'(\psi)\int\left(u^2/2+\mu B\right)\sum_\sigma\left\{\left[F_{\sigma 1}^{\mathrm{sw}}(\nabla_{\mathbf{R}_\perp/\varepsilon_\sigma}\langle\phi_{\sigma 1}^{\mathrm{sw}}\rangle\times\hat{\mathbf{b}})\cdot\nabla_{\mathbf{R}}\psi\right]^{\mathrm{lw}}\right.\right. \\ + \left.\left.\frac{B}{Z_\sigma\lambda_\sigma}\left\langle\left(\frac{Iu}{B}+\rho\cdot\nabla_{\mathbf{R}}\psi\right)\sum_{\sigma'}\mathscr{T}_{\sigma,0}^{*}C_{\sigma\sigma'}^{(1)\mathrm{lw}}\right\rangle\right\}\mathrm{d}u\mathrm{d}\mu\mathrm{d}\theta\right\rangle_\psi \\ - \left\langle\sum_\sigma\int B\left[F_{\sigma 1}^{\mathrm{sw}}\left(u\hat{\mathbf{b}}\cdot\nabla_{\mathbf{R}}\langle\phi_{\sigma 1}^{\mathrm{sw}}\rangle + \frac{\mu}{B}(\hat{\mathbf{b}}\times\nabla_{\mathbf{R}}B)\cdot\nabla_{\mathbf{R}_\perp/\varepsilon_\sigma}\langle\phi_{\sigma 1}^{\mathrm{sw}}\rangle\right.\right.\right. \\ + \left.\left.\left.\frac{u^2}{B}[\hat{\mathbf{b}}\times(\hat{\mathbf{b}}\cdot\nabla_{\mathbf{R}}\hat{\mathbf{b}})]\cdot\nabla_{\mathbf{R}_\perp/\varepsilon_\sigma}\langle\phi_{\sigma 1}^{\mathrm{sw}}\rangle\right)\right]^{\mathrm{lw}}\mathrm{d}u\mathrm{d}\mu\mathrm{d}\theta\right\rangle_\psi \\ + \left\langle\sum_{\sigma,\sigma'}\frac{\tau_\sigma}{\lambda_\sigma^2}\int B\left(u^2/2+\mu B\right)\left[\left\langle\mathscr{T}_{\sigma,1}^{*}C_{\sigma\sigma'}^{(1)\mathrm{sw}}\right\rangle\right]^{\mathrm{lw}}\mathrm{d}u\mathrm{d}\mu\mathrm{d}\theta\right\rangle_\psi.\end{aligned} \tag{12}$$

The zeroth-order piece of the long-wavelength quasineutrality equation imposes the condition $\sum_\sigma Z_\sigma n_\sigma = 0$ on the lowest order particle densities. On the other hand, we have obtained as a solvability condition of the long-wavelength second-order Fokker-Planck equation a time evolution equation for each function n_σ, (11). Thus, we can deduce a time evolution equation for $\sum_\sigma Z_\sigma n_\sigma$. It is important to find out whether $\partial_t \sum_\sigma Z_\sigma n_\sigma \equiv 0$ automatically or, on the contrary, its fulfillment implies additional constraints on lower-order quantities. With all the ingredients mentioned in this paper, it can be shown rigorously (see reference [4]) that $\partial_t \sum_\sigma Z_\sigma n_\sigma \equiv 0$ automatically, so the tokamak is intrinsically ambipolar even in the presence of microturbulence.

References

[1] F. I. Parra and P. J. Catto, Plasma Phys. Control. Fusion **51**, 095008 (2009).
[2] F. I. Parra and P. J. Catto, Plasma Phys. Control. Fusion **52**, 045004 (2010).
[3] F. I. Parra and I. Calvo, Plasma Phys. Control. Fusion **53**, 045001 (2011).
[4] I. Calvo and F. I. Parra, arXiv:1204.1509v2 [physics.plasm-ph].