Sonic-flow gyrokinetic simulations with a unified treatment of all length scales

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Introduction

Conventional sonic-flow gyrokinetics \cite{1} uses a 2-component potential, $\phi = \phi_0 + \phi_1$, where $\phi_0$ is a specified (not self-consistent) large, long-wavelength potential, and $\phi_1$ is a small, short-wavelength potential. However, ordering the parallel vorticity to be small \cite{2}, $|\nabla \times u|\Omega^{-1} \ll 1$, allows a unified description of large, intermediate and small amplitude flows on large, intermediate and small length scales, respectively. We choose to have a Lagrangian where the fields are in the symplectic part, and, as a result of doing so, it is not obvious how to get explicit equations of motion. We have found a method for dealing with such systems when there is an ordering parameter.

Equations

Our electrostatic, slab single-particle Lagrangian \cite{3} is

$$L_p = \left[ A(R) + v_\| \dot{b} + u \right] \cdot \dot{R} + \mu \dot{\theta} - \left( \frac{1}{2} v_\|^2 + \mu B + \frac{1}{2} u^2 + \langle \phi \rangle \right), \quad (1)$$

where $u = B^{-1} \dot{b} \times \nabla \langle \phi \rangle$ and the sonic-flow terms are the third terms in the brackets and parentheses. Our Vlasov-Poisson system for two-dimensional potential perturbations is

$$f \cdot \dot{R} f_i = 0,$$

$$\dot{R} = u + v_\| \dot{b} + B^{-1} \dot{b} \times \dot{u}_1,$$

$$\dot{v}_\| = 0,$$

$$0 = \int d^6 Z \delta (R + \rho - r) [B_\|^* f + B^{-1} \nabla \cdot f \dot{u}_1], \quad (2)$$

where

$$B_\|^* = \dot{b} \cdot (B + \nabla \times u), \quad (3)$$

$\dot{u}_1 = (\partial_t + u \cdot \nabla) u$ and the terms relating to $\dot{u}_1$ are implicit. Both $\dot{R}$ and the Poisson equation (2) depend on the time derivative of the field. Our linearized Poisson equation goes to $-B^{-2} n_0 \nabla^2 \phi = \delta n$ in the $k_\perp \rho_e \sim \epsilon$ limit. As $u$ appears outside the Hamiltonian (in the symplectic part) (1), implicit time-dependence prevents application of standard direct numerical schemes.
Pull-back to original coordinates

For \( k_\perp \rho_t \sim \varepsilon \), we have the small-flow representation,

\[
x = R + \rho + B^{-2} \nabla \phi(R) + \mathcal{O}(\varepsilon^2)
\]

\[
= R + \rho + \mathcal{O}(\varepsilon),
\]

and the sonic-flow representation,

\[
x = R + \rho + B^{-2} \nabla [\phi(R) - \langle \phi \rangle] + \mathcal{O}(\varepsilon^2)
\]

\[
= R + \rho + \mathcal{O}(\varepsilon^2).
\]

Thus, we see that the sonic-flow representation has a smaller difference between the actual location of the particle \( x \) and the gyroring, \( x' = R + \rho \), which is where \( \langle \phi \rangle \) is evaluated [4].

Distribution functions: small vs. sonic flow

We define a Maxwellian in the small- and sonic-flow gyrokinetic formalisms, and determine the corresponding particle distribution in original coordinates. For \( k_\perp \rho_t \sim \varepsilon \), for small flows we have

\[
f = (2\pi)^{-3/2} \exp(-\frac{1}{2} v^2)
\]

\[
= (2\pi)^{-3/2} \exp[-\frac{1}{2} v^2 - \frac{1}{2} u^2 + \mathcal{O}(\varepsilon^2)]
\]

\[
= (2\pi)^{-3/2} \exp(-\frac{1}{2} v^2 + \mathcal{O}[\varepsilon^2]),
\]

and for sonic flows we have

\[
f = (2\pi)^{-3/2} \exp(-\frac{1}{2} v^2)
\]

\[
= (2\pi)^{-3/2} \exp(-\frac{1}{2} v^2 + \mathcal{O}[\varepsilon^2]).
\]

We find that these distributions are compatible.

Numerical scheme

We choose to use a \( \delta f \) PIC code, as in [5], with \( f = f_0 + \delta f \). As our \( B_\parallel \) is potential-dependent (3), care is needed with initialisation. The evolution of the markers proceeds as follows.

1. Take an initial RK4 step by neglecting the terms involving \( \dot{u}_1 \).

2. Compute a cubic spline representation of \( R(t) \) and \( \delta f(t) \) on the interval \([t, t + \Delta t] \), where \( \Delta t \) is the time step.

3. Compute \( \dot{u}_1 \) via \( R(t), \delta f(t) \rightarrow n(t) \rightarrow \phi(t) \rightarrow u(t) \) and finite differences.
4. Take an RK4 step including all terms by using this estimated value of $\dot{u}_1$.

5. Iterate to desired level of convergence.

Alternatively, a multistep or hybrid method could be considered.

**An example problem**

We may test (Figure 1) our iterative scheme with an equation that is of a similar form to our gyrokinetic system of equations, and permits analytic solution,

$$\dot{y} = e^{-y} + \varepsilon \dot{y}.$$  \hfill (4)

### Figure 1.

Absolute error per unit time, versus timestep $h$, of the augmented RK4 scheme used to solve Equation (4), for $\varepsilon = h$ and $\varepsilon = h^2$. These are plotted in black with the expected scaling shown as a red trace.

### Figure 2.

Kelvin-Helmholtz instability growth-rate spectra, with only the difference between the positive- and negative-shear growth rates plotted.

**Kelvin-Helmholtz instability**

Conventional Kelvin-Helmholtz is symmetric in sign of vorticity, but in extended-MHD, an asymmetry appears [6]. Asymmetry also appears in the sonic-flow gyrokinetic model and is of the same magnitude (Figure 2).

**Conclusions**

A sonic-flow gyrokinetic theory with a unified treatment of all length scales has been numerically implemented. The Vlasov-Poisson system (2) is obtained as a whole, directly from the gyrocentre Lagrangian (1), and corresponds to the Hasegawa-Mima equation [7] in the small-flow and $k \perp \rho_t \ll 1$ limits. We use an iterative numerical solution of our Vlasov-Poisson system (2), and this iterative scheme may have general applications. We see sonic-flow symmetry-breaking...
that depends on the sign of $\dot{b} \cdot \nabla \times u$ (Figures 2 and 3). Code verification has been performed with basic slab instabilities. The Poisson solver (using the same numerical scheme [8] as the ORB5 code) is capable of solving 3D global tokamak geometry but is used here for slab and cylindrical cases.

Figure 3. Small- (top) and sonic- (bottom) flow blob propagation: sonic-flow blobs exhibit a shift in the rotation frequency of the vortices that depends on the sign of $\dot{b} \cdot \nabla \times u$.


