A new analytic solution to the collision-free plasma equation with warm ions

N. Jelić[1,2], L. Kos[1], S. Kuhn[2]

[1] *Faculty of Mechanical Engineering, University of Ljubljana, SI-1000 Ljubljana, Slovenia,*

[2] *Institute for Theoretical Physics, University of Innsbruck, A-6020 Innsbruck, Austria*

Determining plasma flow to boundaries in cases when ion mean energies are comparable to, higher than or even considerably above the temperature of the main/bulk electrons (usually Maxwellian) is a fundamental problem of plasma physics which, however, has turned out to be extremely demanding to deal with, even when quite elementary physical and geometric assumptions and simplifications were made. Here we present this problem starting from the one-dimensional time-independent Boltzmann equation for the ion velocity distribution function (VDF) $f_i(x,v)$ in plane-parallel geometry, $v\frac{\partial f_i}{\partial x} - \frac{e}{m_i}\frac{d\Phi(x)}{dx}\frac{\partial f_i}{\partial v} = S(v^2,\Phi)$, coupled with the Poisson equation $\frac{d^2\Phi}{dx^2} = \frac{e}{\epsilon_0}(n_i - n_e)$. This system is to be solved in the plasma approximation, i.e., assuming strict quasineutrality $n_i = n_e$, under symmetric boundary conditions holding at two perfectly absorbing co-planar plates which are located at positions $x = \pm L$ and biased at the self-established electric potential $\Phi(\pm L) \equiv \Phi_W$. The electric potential $\Phi(x)$ is assumed to be monotonically decreasing in both directions from the point (plane) of symmetry satisfying $x = 0$, $\Phi(0) = 0$. The ion density $n_i = \int f_i\, dv$ must equal the electron Boltzmann distributed density $n_e = n_{eo}\mathrm{e}^{e\Phi/kT_e}$ density, while the ion directional velocity, temperature, energy and heat flux have to be calculated as the mth moments of the ion VDF, $\left\langle v_i^m \right\rangle = \int f_i v^m dv/n_i$.

It is convenient to normalize physical quantities as $\frac{x}{L} \leftrightarrow x$, $\frac{n_{i,e}}{n_0} \leftrightarrow n_{i,e}$, $\frac{v_i}{c_{se}} \leftrightarrow v_i$, $\frac{u_i}{c_{se}} \leftrightarrow u_i$, $\frac{T_i}{T_e} \leftrightarrow T_i$, $\frac{e\Phi}{kT_e} \leftrightarrow \Phi = -\varphi$, $\frac{c_{se}f_i}{n_0} \leftrightarrow f_i$, $\frac{LE}{kT_e/e} \leftrightarrow E$ and $\frac{LS(v^2,\varphi)}{n_0} \leftrightarrow S(v^2,\varphi)$, with e the positive elementary charge, k the Boltzmann constant, $c_{se} \equiv (kT_e/m_i)^{1/2}$, m_i the ion mass and $E = -d\Phi/dx \leftrightarrow d\varphi/dx$ the electric field. For completeness we also introduce the smallness parameter $\varepsilon \equiv \lambda_D/L$, where $\lambda_D = (\epsilon_0 kT_{e0}/n_0e^2)^{1/2}$ is the electron Debye length, ϵ_0 is the "vacuum permeability", the vanishing of which is equivalent to the assumption of strict quasineutrality. The ion source is modeled here according to the Bissell and Johnson assumptions [1] $S_i = Rn_n n_{e0}\mathrm{e}^{\beta\Phi/kT_e}\mathrm{e}^{-m_iv^2/2kT_n}/(2\pi kT_n)^{1/2}$, with the ionization rate there proportional to $\sim n_e^{\beta} = \mathrm{e}^{\beta\Phi/kT_e}$ (cf. Harrison and Thompson [2]). Factor β can take arbitrary values but here only the values $\beta = 1$ (after Bissell and Johnson [1]) and $\beta = 0$ (after Scheuer and Emmert [3]) are considered. The term Rn_n [1] refers to the ion creation rate with frequency $\nu_i = Rn_n = c_{se}/L_i$ due to either ionization of gas between the plates or due to inflow from a virtual perpendicular direction. The latter scenario corresponds to application of the scrape-off-layer (SOL) [4] plasma model with the warm ions originating from the plasma-core of a tokamak device. Note that we here strictly distinguish between the source (neutral) temperature T_n and the self-consistently established ion temperature T_i.

With these assumptions and definitions, the Boltzmann equation and its formal solution take the respective forms

$$v\frac{\partial f_i}{\partial x}+E\frac{\partial f_i}{\partial v}=S(v^2,\varphi) \text{ and } f_i(v,x)\rightarrow f_i(v^2/2-\varphi)=2\int_0^{\varphi_W}\frac{d\varphi'}{E(\varphi')}\frac{S(\varphi'-\varphi+v^2/2,\varphi')}{\sqrt{2(\varphi'-\varphi+v^2/2)}},$$

where the electric field must be found from the quasineutrality condition $\int f_i dv = \mathrm{e}^{-\varphi}$. Inserting the Maxwellian source into this condition and integrating over velocity space yields the plasma equation and the ion VDF in the forms

$$\frac{L/L_i}{\sqrt{2\pi T_n}}\int_0^{\varphi_b}\frac{d\varphi'\mathrm{e}^{-\beta\varphi'}\mathrm{e}^{\frac{\varphi'-\varphi}{2T_n}}}{E(\varphi')}\mathrm{K}_0\left(\frac{|\varphi'-\varphi|}{2T_n}\right)=\mathrm{e}^{-\varphi},\;\; f_i=\frac{L/L_i}{\sqrt{2\pi T_n}}\int_{\varphi'}\frac{d\varphi'\mathrm{e}^{-\beta\varphi'}}{\sqrt{2}E(\varphi')}\frac{\mathrm{e}^{\frac{\varphi'-\varphi+y}{T_n}}}{\sqrt{(\varphi'-\varphi+y)}},$$

respectively, there, $y=v^2/2$, K_0 is the Bessel function of zeroth order, φ_b stands for the boundaries of integration, which in the present case $\varepsilon=0$ correspond to the strictly quasineutral plasma edge (φ_{PE}), and is $\frac{L}{L_i}=\mathrm{e}^{-\varphi_W}\sqrt{\frac{m_i}{2\pi m_e}}\left(\int_0^{\varphi_W}\frac{\mathrm{e}^{-\varphi}d\varphi}{E(\varphi)}\right)^{-1}$ as so-called "ionization length", with the numerical values complemented by our fitting formula $L_i/L\simeq\left(\sqrt{2\pi T_n+2}-1+\beta\right)/\pi^2$ presented in Ref. [5]. The domain of integration, symbolically represented by $\int_{\varphi'}$, will be specified below. Note that L/L_i can be manipulated by changing the real or simulated lengths and properly renormalizing the electric field.

Thus the key quantity for calculating the ion VDF and its moment turns out to be the electric field E as a function of the potential. Since in the quasi-neutral plasma E is rather small and, moreover, does not depend strongly on either potential φ or position x, calculating the ion VDF as a function of φ via $E(\varphi)$ as the main physical and mathematical quantity (appearing in the unknown kernel of a complicated integral equation) would require extreme efforts. Nearly complete numerical solutions of the present problem, to be referred to as the Bissell and Johnson (B&J) model, have been obtained with varied temperatures in a series of exhaustive works co-authored by the present authors. However, we are aware that other authors can just *employ* these results (e.g., in tabulated form, which we use to name "empirical") but cannot easily *reproduce* them in any numerical or theoretical form convenient for application to further research. Hence, we have propose an analytic solution to the B&J model, starting from the *initial* assumption that $2T_n$ is fairly well above the maximum possible value of $|\varphi'-\varphi|$, such that approximation $\mathrm{K}_0\left(\frac{|\varphi'-\varphi|}{2T_n}\right)\simeq\ln\left(\frac{4T_n}{\gamma_E|\varphi-\varphi'|}\right)$ (where $\gamma_E=\exp(C_E)=1.78107$ and $C_E=0.57721...$ is the Euler-Mascheroni constant) holds. Then the above exact quasineutrality condition takes the form $\frac{L/L_i}{\sqrt{2\pi T_n}}\int_0^{\varphi_b}\frac{d\varphi'\mathrm{e}^{-\beta\varphi'}}{E(\varphi')}\mathrm{e}^{\frac{\varphi'-\varphi}{2T_n}}\ln\left(\frac{4T_n}{\gamma_E|\varphi-\varphi'|}\right)=\mathrm{e}^{-\varphi}$, which can be instantly solved via applying the Carleman inversion. This yields the electric field

$$\frac{L/L_i}{E(\varphi)}=\frac{2T_n+1}{\sqrt{2\pi T_n}}\frac{\mathrm{e}^{-\frac{a\varphi_b}{2}}\mathrm{e}^{(\beta-1)\varphi}}{\sqrt{\varphi(\varphi_b-\varphi)}}\mathrm{e}^{a\varphi}F(\varphi,T_n)\,, \tag{1}$$

where for brevity $F(\varphi,T_n)\equiv\frac{1}{\pi\,\mathrm{e}^{-a\frac{\varphi_b}{2}}}⨍_0^{\varphi_b}\frac{\sqrt{t(\varphi_b-t)}}{t-\varphi}\mathrm{e}^{at}\,dt+\frac{1}{a\ln F_{Tn}}\mathrm{I}_0\left(a\frac{\varphi_b}{2}\right)$ and $a\equiv a(T_n)=1+\frac{1}{2T_n}$. The symbol $⨍_0^{\varphi_b}$ denotes the principal value, $\mathrm{I}_0(z)$ stands for the zero-order Bessel function, and $F_{Tn}\equiv\frac{16T_n}{\gamma_E\varphi_b}$. It has been shown in Ref. [6] that the solution Eq. (1) perfectly fits the exact numerical results for $T_n \geq 3$, while for lower temperatures, $T_n < 1$, the present quasi-analytic results considerably deviate from exact ones.

When applying the numerical method, a breaking solution appears for a certain potential $\varphi_b(T_n)$, for which the electric field diverges. The theoretical approach, in contrast, consists in finding $\varphi_b(T_n)$ from the condition $F(\varphi_b, T_n) = 0$ and inserting the resulting value into Eq. (1) for obtaining $1/E(\varphi, T_n)$. For this purpose we note that the first term in $F(\varphi, T_n)$ at the points $\varphi = 0$ and $\varphi = \varphi_b$ reduces to $\frac{\varphi_b}{2} \mathrm{I}_1\left(a\frac{\varphi_b}{2}\right) \pm \frac{\varphi_b}{2} \mathrm{I}_0\left(a\frac{\varphi_b}{2}\right)$, where the positive and negative signs correspond to the values $\varphi = 0$ and $\varphi = \varphi_b$, respectively, and $\mathrm{I}_1(z)$ is the first-order Bessel function of the argument z. We denote $F(\varphi, T_n)$ at the points $\varphi = 0$ and $\varphi = \varphi_b$ as $F^{\pm} = \frac{\varphi_b}{2}\left[\left(\frac{2}{a\varphi_b \ln F_{Tn}} \pm 1\right)\mathrm{I}_0\left(a\frac{\varphi_b}{2}\right) + \mathrm{I}_1\left(a\frac{\varphi_b}{2}\right)\right]$, respectively.

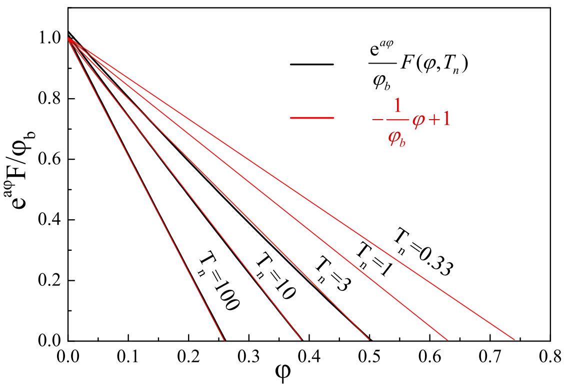

Figure 1: Checking the approximation $\frac{e^{a\varphi}}{\varphi_b}F(\varphi, T_n) \simeq -\frac{\varphi}{\varphi_b} + 1$ for illustrative values of T_n.

For realistic T_n, the plasma potential is limited between $0 < \varphi < \varphi_b(T_n)$, where $\sim 0.25 \simeq \varphi_b(100) < \varphi_b(T_n) < \varphi_b(0) = 0.854$ (see the exact results, e.g., in Ref. [5] or use our approximate formula $\varphi_{b0} = \varphi_{PE} \simeq 1/\ln(\sqrt{\pi}T_n^{3/4} + \pi))$, while $a \equiv a(T_n) = 1 + \frac{1}{2T_n}$ in the range of validity of Eq. (1) is slightly above unity, so that the argument of the Bessel functions, $a\varphi_b/2$, is well below unity and thus $\mathrm{I}_0\left(a\frac{\varphi_b}{2}\right) \simeq 1$, $\mathrm{I}_1\left(a\frac{\varphi_b}{2}\right) \simeq a\frac{\varphi_b}{4}$. The requirement that $F(\varphi, T_n)$ vanishes at φ_b can be satisfied for $F^- = \frac{\varphi_b}{2}\left[\left(\frac{2}{a\varphi_b \ln F_{Tn}} - 1\right) + \frac{a\varphi_b}{4}\right] = 0$. Here we note that it can be easily solved as a quadratic (rather than transcendental) equation, yielding very precise values of $\varphi_b(Tn)$ provided that the approximate $\varphi_{b0}(T_n)$ is inserted into $\ln F_{Tn}$. More important here is the fact that from $F^- = 0$ we find $\frac{2}{a\varphi_b \ln F_{Tn}} = 1 - \frac{a\varphi_b}{4}$, so that inserting this into F^+ we get $F^+ = \varphi_b$ as the upper limit of $F(\varphi, T_n)$. Hence, the approximation $F(\varphi, T_n) \simeq -\varphi + \varphi_b$ is a plausible one, but we find that a perfect one for $T_n > 3$ requires the multiplicative factor $e^{a\varphi}$, cf. Fig. 1, where we plot $e^{a\varphi}F(\varphi, T_n)/\varphi_b$ in comparison with $-\varphi/\varphi_b + 1$ for several temperatures $T_n \geq 3$. Since for lower temperatures the solution Eq. (1) diverges we just illustrate the behavior of the new approximation $-\varphi/\varphi_b - 1$ and suppose that it holds for any sufficiently high T_n. Equation (1) with $e^{a\varphi}F(\varphi, T_n) \simeq -\varphi + \varphi_b$ thus takes the form

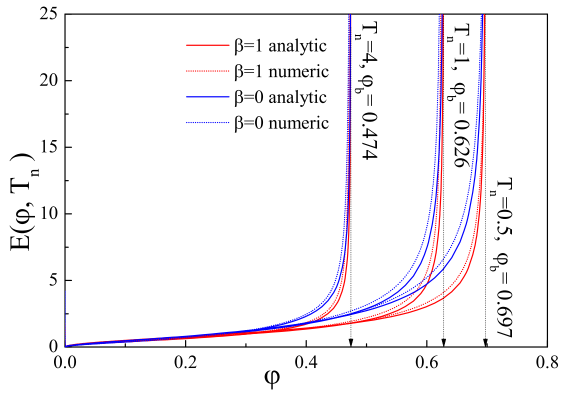

Figure 2: The electric-field profiles

$$\frac{L_i/L}{E(\varphi)} = A_{Tn} e^{(\beta-1)\varphi} \frac{\sqrt{\varphi_b - \varphi}}{\sqrt{\varphi}}, \quad \text{with} \quad A_{Tn} \equiv \frac{2T_n + 1}{\sqrt{2\pi T_n}} e^{-(1+\frac{1}{2T_n})\frac{\varphi_b}{2}}. \tag{2}$$

At this point we recall that in our previous works (cf. Refs.[7, 5] and references

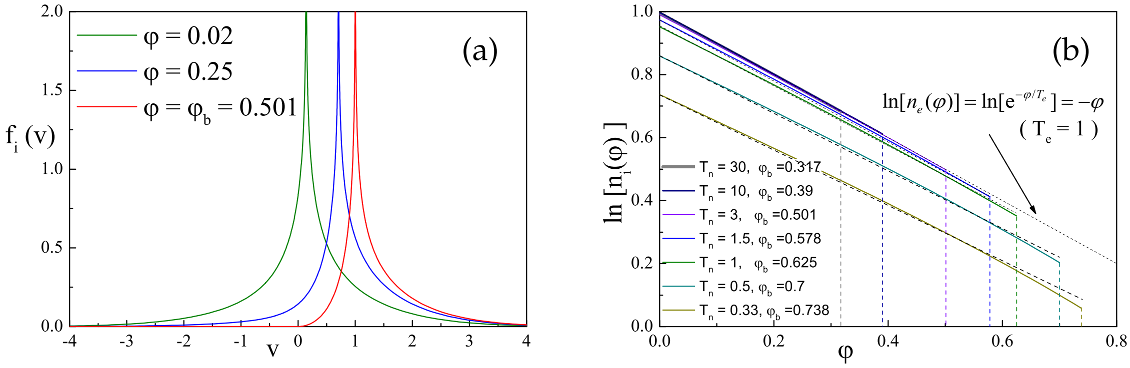

Figure 3: Ion VDFs for $T_n = 3$ at several positions/potentials (a) and ion densities obtained from $n = \int f(v)dv$ for several source temperatures, as plotted in logarithmic representation, in comparison with the Boltzmann electron density (b).

therein) we used instead of A_{Tn} a simple 'ad hoc' approximation ($A_{Tn} \simeq 27\sqrt{T_n}/40$), the relevancy of which can now be easily justified. The second improvement here is the extension from $\beta = 1$ to arbitrary profiles $0 \leq \beta \leq 1$ entering through the multiplicative function $\mathrm{e}^{(\beta-1)\varphi}$.

In Fig. 2 we compare the approximate analytic electric fields as given by Eq. (2) for three temperatures, with the numerical ones considered as the "exact" ones. In either case the unique plasma-edge potential $\varphi(T_n)$ as obtained from the numerical method is used. The cases with exponentially decreasing ($\beta = 1$) and the "flat" ($\beta = 0$) source profiles are presented. A relatively small deviation of the analytic profiles from the numerical ones can be observed.

Introducing the inverse electric field Eq. (2) into the formal solution of the Boltzmann equation, we obtain our ion VDF in the form

$$f_i(v,\varphi) = \frac{(2T_n+1)\mathrm{e}^{-(1+\frac{1}{2T_n})\frac{\varphi_b}{2}}}{2\pi T_n} \int_{\varphi'} \frac{\sqrt{\varphi_b - \varphi'}}{\sqrt{\varphi'}\,\sqrt{2(\varphi' - \varphi + v^2/2)}} \mathrm{e}^{-\varphi' + \frac{\varphi' - \varphi + v^2/2}{T_n}} d\varphi' \,. \qquad (3)$$

The operator $\int_{\varphi'} d\varphi'$ stands for five integrals, acting on the same function of argument φ', i.e., in the operator form: $\int_{\varphi'} d\varphi' = h(-v)\int_{\varphi}^{\varphi_b} d\varphi' + h(-v)h(\sqrt{2\varphi} - v)\left(\int_{\varphi-\frac{v^2}{2}}^{\varphi_b} + \int_{\varphi-\frac{v^2}{2}}^{\varphi}\right)d\varphi' +$ $h(v - \sqrt{2\varphi})\left(\int_0^{\varphi_b} + \int_0^{\varphi}\right)d\varphi'$, where $h(x)$ is the Heaviside step function of argument x. The plasma-edge potential has to be determined with one of the methods referred to above.

In Fig. 3(a) we just illustrate the VDFs as obtained from integration of Eq. (3) for $T_n = 3$ at several positions/potentials. More important results, however, are the ion densities $n = \int f(v)dv$ for several source temperatures, as plotted in logarithmic representation in comparison with the Boltzmann electron density in Fig. 3(a). As is obvious, the quasineutrality is "perfect" (below a few of percent) for $T_n > 1.5$, while for lower T_n the functional dependence $n_i(\varphi)$ starts to deviate from $\mathrm{e}^{-\varphi}$ and the departure from normalization $n_i(0) = \int f_i(v,\varphi)dv = 1$ for $T_n < 1.5$ becomes evident.

References

[1] R. C. Bissell and P. C. Johnson, Phys. Fluids **30**, 779 (1987).
[2] E. R. Harrison and W. B. Thompson, Proc. Phys. Soc. **74**, 145 (1959).
[3] J. T. Scheuer and G. A. Emmert, Phys. Fluids **31**, 3645 (1988).
[4] P. C. Stangeby, *The Plasma Boundary of Magnetic Fusion Devices* (Institute of Physics Publishing Ltd, Bristol, 2000).
[5] L. Kos, N. Jelić, T. Gyergyek, S. Kuhn, and D. D. Tskhakaya, AIP Advances **8**, 105311 (2018).
[6] D. D. Tskhakaya, L. Kos, and N. Jelić, Phys. Plasmas **21**, 073503 (2014).
[7] L. Kos, N. Jelić, S. Kuhn, and D. D. Tskhakaya, Phys. Plasmas **25**, 043509 (2018).