**Strong-flow gyrokinetic simulations with a unified treatment of all length scales**

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**Introduction**

The gyrokinetic ordering parameter was originally[1]

$$\epsilon \sim \omega \Omega^{-1} \sim k_\parallel \rho_t \sim \rho_t L_B^{-1} \sim q\phi T^{-1} \ll 1, k_\perp \rho_t \sim 1,$$

(1)

where $\omega$ is the characteristic fluctuation frequency, $\Omega$ is the gyrofrequency, $k_\parallel$ is the characteristic fluctuation parallel wavenumber, $\rho_t$ is the thermal gyroradius, $L_B$ is the magnetic field length scale, $q$ is the particle charge, $\phi$ is the electrostatic potential, $T$ is the temperature and $k_\perp$ is the characteristic fluctuation perpendicular wavenumber. Ordering (1) can be generalised[2] to give a weak-flow gyrokinetic ordering parameter,

$$\epsilon \sim \omega \Omega^{-1} \sim k_\parallel \rho_t \sim \rho_t L_B^{-1} \sim u v_t^{-1} \ll 1,$$

(2)

where $u$ is the $E \times B$ drift speed associated with $\phi$ and $v_t$ is the thermal particle speed. Ordering (2) cannot be applied to all modern tokamak plasmas in general due to the presence of large flows. A further generalisation[3] gives a strong-flow gyrokinetic ordering,

$$\epsilon \sim \omega \Omega^{-1} \sim k_\parallel \rho_t \sim \rho_t L_B^{-1} \sim u' \Omega^{-1} \ll 1,$$

(3)

where $u'$ is the magnitude of the spatial derivatives of the $E \times B$ drift velocity associated with $\phi$. Using Ordering (3), we present the discretisation of our manifestly conservative Vlasov-Poisson system that is obtained directly from our gyrocentre Lagrangian[4].

**Lagrangian**

Our gyrocentre Lagrangian for electrostatic potential perturbations in slab magnetic geometry up to first order is

$$\Gamma = [A(R) + v_\parallel \mathbf{\hat{b}} + \mathbf{u}] \cdot dR + \mu d\theta - \frac{1}{2} \mathbf{u}^2 - \mu B + \frac{1}{2} \mu^2 + \langle \phi \rangle d\tau,$$

(4)

where we use units such that $m = q = T = 1$, $m$ is the particle mass, $T$ is the temperature, $q$ is the particle charge, $A$ is the magnetic vector potential, $R$ is the gyrocentre position, $\mathbf{\hat{b}}$ is the magnetic field unit vector, $\mathbf{u} = B^{-1} \mathbf{\hat{b}} \times \nabla \langle \phi \rangle$, $\langle \psi \rangle(R, \mu, \tau) = (2\pi)^{-1} \int d\theta d^3r \delta(R + \rho - r) \psi(r, \tau)$.
for any function $\psi$, $\mu$ is the conserved magnetic moment, $t$ is time, $\theta$ is the gyroangle, $r$ is the particle position and $\rho$ is the gyroradius.

This Lagrangian (4) does not contain any $O(\varepsilon)$ terms, as these have all been Lie-transformed to higher order.

**Vlasov-Poisson system**

The Vlasov-Poisson system obtained from our Lagrangian (4) is then

$$\begin{align*}
F_i + \dot{R}_i F = 0,
\dot{R} = u + B^\perp \hat{b} \times \dot{u}_1,
\dot{v}_\parallel = 0,
0 &= \int d^6z \delta(\mathbf{R} + \mathbf{r} - \mathbf{R}) [B^\parallel F + B^{-1} \hat{b} \cdot \nabla \times F (\hat{b} \times \mathbf{u}_1)],
\end{align*}$$

(5)

where $F$ is the distribution function that transforms as a scalar, $F_{,\theta} = 0$, $\Psi,_{\alpha} = \partial_{\alpha} \Psi$ for any quantity $\Psi$ and any coordinate $\alpha$, $i \in \{1, 2, 3\}$,

$$B^\parallel = \hat{b} \cdot (\mathbf{B} + \nabla \times u),$$

(6)

$\dot{u}_1 = (\partial_t + \mathbf{u} \cdot \nabla) \mathbf{u}$ and $Z = (\mathbf{R}, \mu, v_\parallel, \theta)$.

**Numerical Scheme**

The second term in $\dot{R}$ (5) and the second term in the square brackets in our Poisson equation (5) contain a partial time derivative of the potential-dependent flow velocity. We use that these terms are one order smaller in $\varepsilon$ than their neighbouring terms in order to facilitate the numerical solution of our Vlasov-Poisson system (5). We choose to solve our Vlasov-Poisson equations (5) using a $\delta f$ particle-in-cell (PIC) method, as in [5]. That is to say, we use Monte Carlo markers to represent distribution function quanta that are evolved according to consistent fields, whilst employing the splitting

$$F = F_0 + \delta F,$$

where we choose the equilibrium part $F_0 = n_0(2\pi T)^{-\frac{3}{2}} e^{-\frac{1}{2}v^2/T}$ to be Maxwellian, $n_0(\mathbf{R})$ is the background density, $T(\mathbf{R})$ is the temperature, $v$ is the particle velocity and the fluctuating part $\delta F$ is discretised as follows. We define

$$\delta F = N_p N^{-1} \sum_{n=1}^{N} 2\pi B^\parallel^{-1} w_n(t) \delta(\mathbf{R} - \mathbf{R}_n(t)) \delta(\mu - \mu_n(t)) \delta(v_\parallel - v_\parallel n(t)),$$

where $N_p$ is the number of particles, $N$ is the number of markers,

$$w_n = \delta F_n V_{pn}$$

(7)
is the marker weight, $\delta F_n$ is the average value of $\delta F$ in the marker phase space volume

$$V_{pn} = d^6z_n dN_n^{-1}, \quad (8)$$

d$^6z_n = B_\parallel^* d^3R d\mu dv_\parallel d\theta$ is an infinitesimal,

$$dN_n = NN_p^{-1} f_m(R, v_\perp, v_\parallel) d^3R v_\perp dv_\perp dv_\parallel d\theta$$

is the number of markers in $d^6z_n$ and $f_m$ is an arbitrary probability density function that transforms as a scalar density. Since our $B_\parallel^*$ is potential-dependent (6), we initially use $B_\parallel^* = B$ when computing $V_{pn}$ (8). However, upon computing the potential, we may compute $B_\parallel^*$ correctly (6), and thus correct $V_{pn}$. Additionally, we may choose to keep $\phi$ constant when correcting $V_{pn}$: in order to do this, we keep $w_n$ constant and adjust $\delta F_n$ using Equation 7. The only consequence of doing this is that initialisation with a particular distribution function is hindered. We may initially neglect the second term in the square brackets in our Poisson equation (5). We may approximate $\dot{u}_1$ with a forward difference by taking a time step using $\dot{R} = u$.

**Slab results**

In the case of a weak-flow, each interacting vortex pulls the other around its centre, resulting in propagation. In the case of a strong-flow, there is a shift in the rotation frequency of the vortices that depends on the sign of the vorticity.

Fig. 3. Comparison of weak- (top) and strong- (bottom) flow blob propagation, where we have used periodic boundary conditions.

**Stand-alone Poisson solver**

We have developed a stand-alone Poisson solver with the following features: arbitrary-wavelength perturbations; cubic B-spline finite-element discretisation; slab and cylindrical geometries; background density and temperature gradients; MPI parallelisation; Fortran source code; based on the solver from the ORB5 code [6]. The following features are planned: extension to three dimensions; field-aligned geometry; nonlinear solution via a multigrid approach.
Solution of our Poisson equation

Due to the presence of $\phi$ in the symplectic part of our Lagrangian (4), $\mathbf{u}_1$ (5) is present in our Poisson equation (5). Formally, we may solve our Vlasov-Poisson system by writing our system Lagrangian $L_s$ is terms of our particle Lagrangian $L_p$ as

$$L_s = \int dZ F L_p[Z, \dot{Z}, \phi(Z, \dot{Z})],$$

where species and temporal subscripts have been suppressed. Whilst a formal solution for $\dot{Z}$ and $\phi$ exists, it is computationally intractable due to the dependence of $\phi$ on $\dot{Z}$ (5). We may use that the second term in the square brackets in our Poisson equation (5) is one order smaller in $\varepsilon$ than the first term, and perform an iterative solution of our Vlasov-Poisson system (5).

Conclusions and future work

A strong-flow gyrokinetic theory with a unified treatment of all length scales has been numerically implemented. Our Vlasov-Poisson system (5) is manifestly conservative, as it is obtained as a whole, directly from our Lagrangian (4). We use an iterative numerical solution of our Vlasov-Poisson system (5). We see strong-flow symmetry-breaking that depends on the sign of $B^*_{\parallel}$ (6). Code verification has been performed with basic slab instabilities. An arbitrary-wavelength, stand-alone Poisson solver has been developed based on the ORB5 code [6]. The manifest conservation of our Vlasov-Poisson system (5) is preserved with our finite-element discretisation.

Centrifugal and drift instability simulations are to be performed. An ultimate goal would be a general magnetic geometry, electromagnetic numerical implementation.

References


