Ponderomotive force in a travelling wave

R.A. Cairns[1], D.A. Burton[2], B. Ersfeld[3], A. Noble[3], S. Yoffe[3], D.A. Jaroszynski[3],

[1] *University of St Andrews, St Andrews, UK*

[2] *Lancaster University, Lancaster, UK*

[3] *University of Strathclyde, Glasgow, UK*

The ponderomotive force is a widely used concept in plasma physics, particularly in the field of laser-plasma interactions. The familiar, widely used expression for this force says that in an oscillating electromagnetic field where the electric field amplitude is $E_0(\boldsymbol{r})$ a particle of mass m and charge q feels an average force

$$\mathbf{F} = -\frac{q^2}{4m\omega^2}\nabla E_0^2 \tag{1}$$

with ω the frequency of the field. An essential assumption is that the scale length of the amplitude variation is long compared to the amplitude of the high frequency oscillation of the particle.

Taking the oscillating field to go as $\cos(\boldsymbol{k}.\boldsymbol{r} - \omega t)$ a key step in obtaining this formula is to assume that the oscillation displacement about some point $\boldsymbol{r}_0$ is just

$$\xi = -\frac{qE_0(\boldsymbol{r}_0)}{m\omega^2}\cos(\boldsymbol{k}.\boldsymbol{r}_0 - \omega t) \tag{2}$$

in the direction of the electric field. The calculation is non-relativistic, so any magnetic field force is neglected. Also neglected is the contribution of the perturbation to the term $\boldsymbol{k}.\boldsymbol{r}$ in the phase, raising the question of how the force is modified in a travelling wave.

Over the years much effort has been expended in obtaining more accurate expressions for the ponderomotive force. Here we will only have space to refer to a few key papers directly relevant to our work, but extensive references to other work can be found in these. We compare the results of theory with numerical solutions of the exact equations for the particle orbits and show that the theory does not always give the correct behaviour in physically important scenarios. We begin with a summary of work on the problem of a purely electrostatic wave, discussed in more detail elsewhere [1], then look at the problem of relativistic motion in an electromagnetic wave, a situation of more practical interest. In both cases we show that the force on a particle may, in some circumstances, change sign along its orbit even though ∇E_0^2 is always in the same direction.

For the electrostatic case we take a field in the x direction given by

$$E(x,t) = E_0(x)\cos(kx - \omega t) \tag{3}$$

so the motion is purely one dimensional. For the purposes of numerical integration of the equation of motion we scale the variables with time in units of ω^{-1}, the electric field in units of $E_0(0)$ and length in terms of $\frac{qE_0(0)}{m\omega^2}$. We shall generally give numerical results for the scaled field going as $E_0(x) = \exp(-x/500)$ though the conclusions are not sensitive to what exactly is chosen. With the field profile specified the only parameter left is the scaled k. As a problem easily related to analytic theory we look at the behaviour of a particle launched from large x with a scaled velocity $-1/\sqrt{2}$. According to (1) this particle should be reflected at $x = 0$ with its incoming orbit exactly reversed. The best theoretical result we have been able to find for comparison is given by Bauer et al [2] who use an averaged Lagrangian technique to obtain an explicit formula for the force. If the average velocity of the particle is V, then in our scaled units their formula is

$$\frac{dV}{dt} = -\frac{1}{4}\frac{(1-kV)(1-3kV)}{(1-kV)^4 - \frac{3}{2}k^2E_0^2}\frac{dE_0^2}{dx}. \tag{4}$$

In obtaining the potential for the averaged Lagrangian the particle displacement around a point x_0 is assumed to be given by

$$\xi = -\frac{qE_0(x_0)}{m(\omega - kV)^2}\cos(kx_0 - \omega t), \tag{5}$$

the Doppler shifted equivalent of (2). Again this neglects the contribution of the oscillation to the kx term in the phase and the question arises as to whether this matters.

Comparing the results of numerical integration of the exact equation of motion of the particle with the predictions of (4) we find very good agreement for scaled values of k up to around 0.45. As is evident from the formula, there is no longer symmetry between the inward and outward legs of the orbit, but a notable feature is that the asymptotic outward velocity is always $1/\sqrt{2}$, exactly reversing the initial velocity. That this is the case for the analytic formula follows from the fact that a time independent Lagrangian L has an invariant

$$V\frac{\partial L}{\partial V} - L \tag{6}$$

which, when the electric field is negligible, is just the particle kinetic energy. This invariant gives a relation between the average velocity and the electric field amplitude that is independent of the spatial variation of the latter, a property we have verified by looking at different profiles numerically. Going to higher values of k we find that (4) breaks down because the denominator becomes singular, while the numerical solution continues to behave well and retains the symmetry of inward and outward asymptotic velocities. A simple argument giving the limit of validity of the formula is that the velocity cannot cross the value $1/(3k)$ while if the Lagrangian averaging is valid it should approach $1/\sqrt{2}$ for large t. These requirements are incompatible if $k > \sqrt{2/3} \approx 0.47$.

If we go to $k = 0.6$ then we obtain the result shown in Figure 1, in which the acceleration is in the direction of ∇E_0^2 along a large part of the outward trajectory. Such behaviour can be obtained from (4) if $1/3 < kV < 1$, described as "uphill acceleration" by Bauer et al, but their equation cannot reproduce the transition from downhill to uphill acceleration along a single particle orbit. Indeed, any averaged Lagrangian in which the potential is quadratic in the field amplitude must, because of the existence of the invariant (6), have a unique value of E_0^2 corresponding to any value of V, while Figure 1 shows the particle to have the same value of V at positions with different field amplitudes. This rules out theories [3, 4] in which the ponderomotive potential is a quadratic involving the plasma susceptibility. Finally we note that such a value of k corresponds to an oscillation amplitude at the origin of around 0.1 of a wavelength so would not appear to be in an impossible parameter range.

While the above shows that, in principle, standard theories of the ponderomotive force can break down, it may be of limited practical importance for laser-driven plasmas. We now show that similar anomalous behaviour can occur in an electromagnetic wave in the relativistic regime with parameters that could realistically apply to the variation of field amplitude along the axis of a focused laser beam. We assume propagation along the x direction with the electric field taking the same form as (3), but in the y direction. From Maxwell's equations the corresponding magnetic field along the z axis is

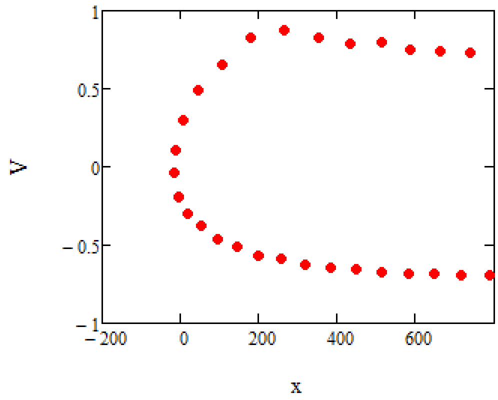

Figure 1: *average velocity as a function of position for k=0.6*

$$B(x,t) = \frac{1}{\omega}\frac{dE_0}{dx}\sin(kx - \omega t) + \frac{kE_0}{\omega}\cos(kx - \omega t) \tag{7}$$

There is now, of course, a new characteristic velocity, the speed of light c. Using the scaling described above the scaled value of c is a^{-1} where

$$a = \frac{qE_0(0)}{m\omega c} \tag{8}$$

is just the familiar dimensionless amplitude used in the study of laser plasmas, evaluated at the origin. For a wave propagating in a plasma the phase velocity is always greater than c so we cannot choose arbitrary values of k. To obtain realistic parameters we instead specify v_{ph}/c and the scaled value of k is then ac/v_{ph}. For the numerical calculation we used the same initial condition as before so that in the non-relativistic regime and small k we expect simple reflection at the origin. In Figure 2 we show the averaged momentum in the x direction for $a = 10$, $v_{ph}/c = 1.05$. As for the electrostatic wave we see that there is a region of uphill force and that along a single particle trajectory there can be a transition from average force down the intensity gradient to average force up the intensity gradient.

In conclusion, we have demonstrated, via direct numerical integration of particle orbits, that for both electrostatic and electromagnetic waves there can be physically realistic conditions under which the ponderomotive force on a particle is in the opposite direction to what would normally be expected. While such an uphill force can be obtained from (4), the transition from normal downhill force to uphill force seen in our results cannot. Indeed, no Lagrangian obtained from a potential quadratic in the field can show this behaviour, suggesting that an appropriate theory would need to go to higher order in the field amplitude. That the system is still described by a suitable Lagrangian is suggested by the fact that the relation between average velocity and electric field amplitude, at least in the electrostatic case, has been verified to be the same for different electric field intensity profiles so that there is some invariant independent of the profile. In future work we hope to extend the theory of the ponderomotive force to encompass the regimes we have discovered, give a more detailed account of when anomalous behaviour occurs and explore possible implications of our work for experiments on laser-plasma interactions, particularly at future facilities like ELI.

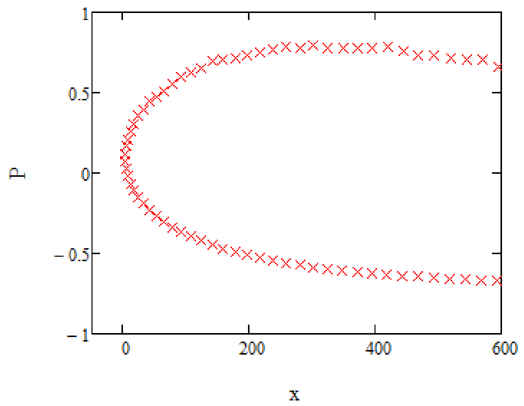

Figure 2: *Average x momentum as a function of x*

Acknowledgements. Support was provided by UK Engineering and Physical Sciences Research Council (EPSRC) grants EP/J018171/1, EP/J500094/1 and EP/N028694/1, and the EC's H2020 LASERLAB-EUROPE (grant no. 654148) and EuPRAXIA (grant no. 653782). Figures were produced using ODE routines from Mathcad 15 and can readily be reproduced with any standard ODE solver.

References

[1] D. A. Burton, R. A. Cairns, B. Ersfeld, A. Noble, S. Yoffe, D. A. Jaroszynski, Proc. SPIE 10234, Relativistic Plasma Waves and Particle Beams as Coherent and Incoherent Radiation Sources II, 102340G (2017); doi: 10.1117/12.2270542

[2] D. Bauer, P. Mulser and W.H. Steeb, Phys. Rev. Lett **75**, 4622 (1995)

[3] J.R. Cary and A.N. Kaufman, Phys. Rev. Lett **39**, 402 (1977)

[4] I.Y. Dodin and N.J. Fisch Phys. Rev. E **77**, 036402 (2008)