Properties of the MHD force operator in the presence of a resistive wall

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1. Introduction. It is known [1-4] that in the ideal magnetohydrodynamics (MHD) the normal modes are either purely oscillating or purely growing/damped due to the self-adjointness of the ideal MHD force operator. This fact was proved in [1-4] for the plasma surrounded by an ideally conducting wall. Our aim is to investigate the properties of this operator in the presence of a resistive wall, so that the boundary conditions will be the main factor affecting the result. We do it by following the method described in [4], but now with the energy dissipation in the resistive wall which is the main difference of our work from [4].

2. Derivations. A static (with mass velocity \( V_0 = 0 \)) toroidal magnetically confined ideally conducting plasma (its volume is denoted as “\( pl \)”) surrounded at some distance with a resistive wall (with its inner surface “wall-”) is considered. There is a vacuum gap (“\( gap \)”) between the plasma and the wall.

We start from the equation of energy transfer in ideal (no energy dissipation) MHD (for example, equation (5.39) in [3])

\[
\frac{\partial}{\partial t} \left( \frac{\rho V^2}{2} + \frac{p}{\Gamma-1} + \frac{B^2}{2} \right) + \nabla \cdot \left( \frac{\rho V^2 V}{2} + \frac{\Gamma}{\Gamma-1} pV + E \times B \right) = 0, \tag{1}
\]

where \( \rho \) is the plasma density, \( V \) is its mass velocity, \( B(E) \) is the magnetic (electric) field, \( p \) is the plasma pressure, \( j = \nabla \times B \) is the current density, \( \Gamma \) is the ratio of specific heats.

Integration of (1) over the volume enclosed by the toroidal wall yields

\[
\frac{\partial E}{\partial t} = - \oint_{\text{wall}} E \times B \cdot dS, \quad \text{where} \tag{2}
\]

\[
E = \int_{pl+gap} u dV \quad \text{with} \quad u = \frac{\rho V^2}{2} + \frac{p}{\Gamma-1} + \frac{B^2}{2}. \tag{3}
\]

Consider small displacements \( \xi(r_0, t) = r - r_0 \) of the plasma from its equilibrium position \( r_0 \). Then the perturbation of the full energy \( \delta E = E(r_0 + \xi) - E(r_0) \) may be expanded to the second order in \( \xi \) and \( \dot{\xi} \) (hereinafter \( \dot{f} \equiv \frac{\partial f}{\partial t} \)) and be presented as

\[
\delta E(\xi, \dot{\xi}) = K(\dot{\xi}, \dot{\xi}) + M(\ddot{\xi}, \dot{\xi}) + \delta W(\xi, \dot{\xi}), \tag{4}
\]

where

\[
K(\dot{\xi}, \dot{\xi}) = \frac{1}{2} \int_{pl} \rho_0 \dot{\xi}^2 dV, \tag{5}
\]

\[
M(\ddot{\xi}, \dot{\xi}) = \int_{pl} \rho_0 \dot{\xi} \dot{\xi} \ddot{\xi} dV, \tag{6}
\]

\[
\delta W(\xi, \dot{\xi}) = \int_{pl+gap} \left( \frac{\rho V^2}{2} + \frac{p}{\Gamma-1} + \frac{B^2}{2} \right) \xi dV. \tag{7}
\]
\( \delta W(\xi, \hat{\xi}) \) is a functional quadratic in \( \xi \), \( M(\xi, \hat{\xi}) \) is a functional bilinear in \( \xi, \hat{\xi} \), and \( b \) is the perturbation of the magnetic field (the magnetic energy \( 0.5 \int_{\text{gap}} b^2 dV \) is included in \( \delta W(\xi, \hat{\xi}) \)).

The terms linear in \( \hat{\xi} \) do not appear in (4) because \( \nabla E(\mathbf{r}_p) = 0 \) in equilibrium.

According to (2) a relation is valid

\[
\frac{\partial}{\partial t} \delta E(\xi, \hat{\xi}) = -F_w^{-}, \quad \text{where} \quad F_w^{-} = \int_{\text{wall-}} (\mathbf{E}_i \times b) \cdot d\mathbf{S}_w^{-}
\]

(6)

and \( \mathbf{E}_i \) is the electric field perturbation. In the standard stability theory with an ideal wall, \( \mathbf{n}_w \times \mathbf{E}_i = 0 \) and \( F_w^{-} = 0 \) (\( \mathbf{n}_w \) is the unit normal to the inner surface of the wall). We assume the wall resistive. Then \( F_w^{-} \neq 0 \), depending on \( \mathbf{E}_i \) and \( b \) at the wall. These quantities are related to \( \mathbf{E}_i = -\hat{\xi} \times \mathbf{B}_0 \) and \( b = \nabla \times [\hat{\xi} \times \mathbf{B}_0] \) in the plasma through the boundary conditions at the plasma surface: \( \mathbf{n}_{pl} \times \mathbf{E}_i = - (\mathbf{n}_{pl} \cdot \hat{\xi}) \mathbf{B}_0 \) and \( \mathbf{n}_{pl} \cdot (\mathbf{b}_{gap} - \mathbf{b}_{pl}) = 0 \). This coupling has a consequence that \( b = 0 \) everywhere at \( \xi = 0 \), but, maybe, \( \mathbf{b} \neq 0 \), if \( \hat{\xi} \neq 0 \) at this moment. When \( \hat{\xi} = 0 \), we have \( \mathbf{E}_i = 0 \), though, maybe, \( b \neq 0 \). This, in particular, means that \( F_w^{-} = 0 \) at either \( \xi = 0 \) or \( \hat{\xi} = 0 \). We use this property below.

With the usage of (5), the left-hand side of the energy balance (6) can be written in the extended form as

\[
2K(\hat{\xi}, \hat{\xi}) + M(\xi, \hat{\xi}) + M(\hat{\xi}, \hat{\xi}) + \delta W(\xi, \hat{\xi}) + \delta W(\hat{\xi}, \hat{\xi})
\]

(7)

As explained above, it must be zero at either \( \xi = 0 \) or \( \hat{\xi} = 0 \). Therefore, \( M(\hat{\xi}, \hat{\xi}) = M(\xi, \hat{\xi}) = 0 \).

The logic of the proof is the same as in [4], and the presence of \( F_w^{-} \) in (6) does not spoil it. Let us add that \( \xi \) and \( \hat{\xi} \) are related by

\[
\rho_0 \hat{\xi} = F_s(\xi),
\]

(8)

the standard equation of small oscillations that nullifies the first-order variation of (1).

Then relation (6) reduces to

\[
2K(\hat{\xi}, F_s(\xi)/\rho_0) + \delta W(\xi, \hat{\xi}) + \delta W(\hat{\xi}, \hat{\xi}) = \int_{\text{wall-}} (\hat{\mathbf{A}} \times \nabla \times \mathbf{A}) \cdot d\mathbf{S}_w^{-},
\]

(9)

where we introduced the vector-potential by \( \mathbf{E}_i = -\hat{\mathbf{A}} \) so that \( b = \nabla \times \mathbf{A} \).

The combination \( \delta W(\xi, \hat{\xi}) + \delta W(\hat{\xi}, \hat{\xi}) \) in (9) is invariant with respect to the replacement of the arguments \( \xi \) and \( \hat{\xi} \), and, accordingly, \( \mathbf{A} \) and \( \hat{\mathbf{A}} \). Therefore,

\[
2K(\hat{\xi}, F_s(\xi)/\rho_0) - 2K(\xi, F_s(\xi)/\rho_0) = \int_{\text{wall-}} [\hat{\mathbf{A}} \times \nabla \times \mathbf{A} - \mathbf{A} \times \nabla \times \hat{\mathbf{A}}] \cdot d\mathbf{S}_w^{-},
\]

or

\[
\int_{pl} \hat{\xi} \cdot \mathbf{F}_s(dV) - \int_{pl} \xi \cdot \mathbf{F}_s(dV) = \int_{\text{wall-}} (\hat{\mathbf{A}} \times \nabla \times \mathbf{A} - \mathbf{A} \times \nabla \times \hat{\mathbf{A}}) \cdot d\mathbf{S}_w^{-},
\]

(10)
where we have used the consequence of (5) and (8):

\[ 2K(\eta, F_s(\xi)/\rho_0) = \int_{pl} \eta \cdot F_s(\xi) dV. \]

The equality (10) consists of 2 functionals bilinear in \( \xi, \hat{\xi} \) and \( A, \hat{A} \). Thus we may substitute the pair \( \xi, \hat{\xi} \) in (10) for any arbitrary vector fields \( \eta(r,t) \) and \( Q(r,t) \), belonging to the same vector space as \( \xi(r,t) \) with \( A(r,t) \). As a result, we will have

\[ \int_{pl} \eta \cdot F_s(\xi) dV - \int_{pl} \hat{\xi} \cdot F_s(\eta) dV = \int_{wall} (Q \times \nabla \times A - A \times \nabla \times Q) \cdot dS. \quad (11) \]

It can be derived directly from relations (8.43)-(8.44) of [3] setting there \( n \times A \neq 0, n \times Q \neq 0 \). If the wall is ideally conducting, i.e. \( n \times A = 0, n \times Q = 0 \), the right-hand side of (11) is zero. Then (11) gives a conventional result [1-4] – self-adjointness of \( F_s \).

For a resistive wall with \( j_i = \sigma E_i \), equation (11) can be transformed in (\( \sigma \) is the conductivity of the wall)

\[ \int_{pl} \eta \cdot F_s(\xi) dV - \int_{pl} \hat{\xi} \cdot F_s(\eta) dV = \int_{wall} \sigma[(Q \cdot A) - (A \cdot Q)] dV = 0. \quad (12) \]

We can introduce a complex displacement and vector-potential by \( \xi = \xi_R + i\xi_I, A = A_R + iA_I \), and the same for \( \eta \) and \( Q \). We demand that the real and imaginary parts of these complex functions belong to the same vector space as \( \xi \) and \( A \). Substitution of such complex functions in (12) does not violate this equality. This means that (11) and, consequently, (12) are valid for complex vector-functions.

Now, substituting in (12) complex vectors \( \eta = \xi^*, Q = A^* \) with * denoting complex conjugation and assuming the time dependence \( \propto \exp(\gamma t) \) with \( \gamma = \gamma_R + i\gamma_I \), we obtain

\[ \int_{pl} \xi \cdot F_s(\xi) dV - \int_{pl} \hat{\xi} \cdot F_s(\hat{\xi}) dV = -2i\gamma_I \int_{wall} |A|^2 dV. \quad (13) \]

Now let us assume that \( \gamma_I \neq 0 \). \quad (14)

Then it follows for \( \xi \propto \exp(\gamma t) \) from (8) that

\[ \int_{pl} \xi \cdot F_s(\xi) dV - \int_{pl} \hat{\xi} \cdot F_s(\hat{\xi}) dV = 4i\gamma_R \int_{pl} \rho_0 |\xi|^2 dV. \]

This relation along with (13) gives

\[ \gamma_R = -\frac{1}{2} \int_{wall} \sigma |A|^2 dV / \int_{pl} \rho_0 |\xi|^2 dV < 0. \quad (15) \]

This result (“a static ideal toroidal magnetically confined plasma surrounded by a resistive wall is always stabilized”) is absolutely unrealistic. Moreover, for an ideally conducting wall the right hand side of (15) gives a wrong result \( \gamma_R = 0 \) instead of \( \gamma_R \gamma_I = 0 \) [1-3]. It proves that our assumption (14) was wrong and \( \gamma_I = 0 \) for an ideal toroidal plasma surrounded by a resistive wall.
Now it is clear that
\[
\int_{pl} \eta \cdot \mathbf{F}_s(\xi) dV - \int_{pl} \xi \cdot \mathbf{F}_s(\eta) dV = 0, \tag{16}
\]
when \( \mathbf{n}_w \times \mathbf{E}_i \neq \mathbf{0} \) and \( \xi, \eta \propto \exp(\gamma t), \cos((k \cdot r) - \omega t), \sin((k \cdot r) - \omega t) \)
\( \tag{17} \)
It can be also obtained that the growth rate must satisfy the quadratic equation [5]
\[
\gamma^2 \int_{pl} \rho_0 \| \mathbf{A} \|^2 dV + \gamma R \int_{wall} \sigma |\mathbf{A}|^2 dV + C = 0, \text{ where } C \text{ is real.}
\]
A small displacement of the perturbed plasma from its equilibrium trajectory in the presence of its equilibrium rotation with a mass velocity \( \mathbf{V}_0 \) is described by the Frieman-Rotenberg equation [6, 7]
\[
\rho_0 \frac{d^2 \xi}{dt^2} = \mathbf{F}_s(\xi) + \nabla \cdot (\xi \rho_0(\mathbf{V}_0 \cdot \nabla) \mathbf{V}_0) = \mathbf{F}(\xi),
\]
where \( \frac{d}{dt} = \frac{\partial}{\partial t} + (\mathbf{V}_0 \cdot \nabla) \), \( \mathbf{F}_s(\xi) \) is a volume force density without and \( \mathbf{F}(\xi) \) - with the plasma equilibrium rotation. It is easy to check that an equation
\[
\int_{\mathcal{V}_s} \eta \cdot \mathbf{F}(\xi) dV - \int_{\mathcal{V}_s} \xi \cdot \mathbf{F}(\eta) dV = \int_{\mathcal{V}_s} \{ \eta \cdot \nabla \cdot (\xi \rho_0(\mathbf{V}_0 \cdot \nabla) \mathbf{V}_0) - \xi \cdot \nabla \cdot (\eta \rho_0(\mathbf{V}_0 \cdot \nabla) \mathbf{V}_0) \} dV \neq 0
\]
is valid \( (\eta \neq \xi) \) for the perturbation time dependencies listed in (17). This means that equilibrium plasma rotation brings non-self-adjointness into the force operator.

3. Conclusion. It has been proved that the force operator of an ideal plasma surrounded by a resistive wall and displaced from its position of static equilibrium is self-adjoint for most commonly used perturbation time dependencies (17). In a general case, this property of the force operator of a static plasma is determined only by how the perturbation varies in time. The plasma equilibrium rotation that makes the force operator of an ideal plasma explicitly non-self-adjoint is needed to make \( \gamma \) (real for a plasma with \( \mathbf{V}_0 = \mathbf{0} \)) complex.