# Hamiltonian approach for evaluation of toroidal torque from finite amplitude non-axisymmetric perturbations of a tokamak magnetic field in resonant transport regimes

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## Introduction

Neoclassical toroidal viscosity (NTV) [1-3] caused by non-axisymmetric magnetic perturbations has an important influence on plasma stability and transport in tokamaks. In this paper, a method to calculate non-ambipolar radial diffusion coefficients, which determine the NTV torque in resonant transport regimes, is presented. In earlier calculations, these regimes have been shown to be important for ion NTV in ASDEX-Upgrade [4] in the case of medium scale perturbations created by a coil system for ELM mitigation (*resonant magnetic perturbations* or RMPs). For finite perturbation amplitudes, nonlinear attenuation can reduce the torque predicted there by the NEO-2 and NEO-RT (resonant transport) code within quasilinear approximation. To further analyze these effects, the quasilinear description based on Hamiltonian theory using action-angle variables [5, 6] used in the code NEO-RT has been extended to include nonlinear attenuation. The mechanism for this attenuation can be summarized as follows.

In presence of non-axisymmetric magnetic perturbations, additional orbit classes arise, e.g. superbanana orbits where bananas play the role of the guiding center. These orbits are traversed with a nonlinear bounce frequency $\omega_{\mathrm{bN}}$. For the kinetic description at low collisionalities, where $\omega_{\mathrm{b}}$ is much larger than the collision frequency $\nu_{\mathrm{eff}}$, there are two limiting cases: If the nonlinear orbit frequency $\omega_{\mathrm{bN}}$ is much smaller than $\nu_{\mathrm{eff}}$, the nonlinear phase $\phi$ is frequently randomized by collisions and one obtains quasilinear resonant transport regimes. In particular, this is the case for formally infinitesimal perturbation amplitudes. If $\omega_{\mathrm{b}}$ becomes larger than $\nu_{\mathrm{eff}}$, the solution gradually changes to the corresponding nonlinear regime.

The presented method provides a consistent description of these resonant transport regimes in the mentioned limiting cases and the transition region. In addition to superbanana and superbanana plateau regime, the description of general nonlinear drift-orbit resonance regimes is possible without simplification of the plasma geometry.

## Kinetic equation, conservation laws and perturbed Hamiltonian

Using action-angle variables $(\mathbf{J},\theta)$, the kinetic equation for the ion distribution function $f(\mathbf{J},\theta)$ takes the form

$$\frac{\partial f}{\partial t}+\{f,H\}=\hat{L}_C f, \tag{1}$$

with Hamiltonian $H$, Poisson brackets $\{\cdot,\cdot\}$ and momentum conserving collision operator $\hat{L}_C$. From the generalized flux-surface averaged conservation law one obtains the expressions for non-ambipolar radial particle flux and torque density, respectively, with

$$\Gamma_n=\frac{1}{S}\int \mathrm{d}^3\theta \int \mathrm{d}^3 J\,\delta\left(r-r_c\right)\{r_c,H\} f, \qquad T_\varphi^{NA}=-\frac{1}{S}\int \mathrm{d}^3\theta \int \mathrm{d}^3 J\,\delta\left(r-r_c\right)\frac{\partial H}{\partial \varphi_c} f. \tag{2}$$

Here, $S$ is the flux surface area, $r$ is the effective radius, $r_c = r_c(\theta, \mathbf{J}) = r_c(\mathbf{r})$ is the particle radial position expressed via phase space variables and $\varphi_c = \theta^3$ is the toroidal canonical angle. For a non-axisymmetrically perturbed system with $H(\mathbf{J}, \theta) = H_0(\mathbf{J}) + H_1(\mathbf{J}, \theta)$ for a single mode $\mathbf{m}$ in canonical angles, $H_1 = \mathrm{Re}[H_\mathbf{m} \exp(i\mathbf{m} \cdot \theta)]$, a resonant angle $\phi = \mathbf{m} \cdot \theta + \arg H_\mathbf{m}$ is defined, which describes the nonlinear oscillation around the resonance

$$\Omega \equiv \frac{\partial H_0(J)}{\partial J} = \mathbf{m} \cdot \frac{\partial H_0(\mathbf{J})}{\partial \mathbf{J}} = 0, \tag{3}$$

and absorbs the complex phase of $H_\mathbf{m}$. A canonical point transformation replacing the canonical toroidal angle $\theta^3$ by $\phi$ yields new actions

$$J'_k = J_k - \frac{m_k}{m_3} J_3, \qquad k = 1,2; \qquad\qquad J'_3 = J_3/m_3 \equiv J. \tag{4}$$

Omitting the first two actions (they are invariants of motion) in the notation and expanding around the resonance up to quadratic order with $\Delta J := J - J_{\mathrm{res}}$ yields the Hamiltonian of the nonlinear pendulum,

$$H(\phi, J) = \frac{1}{2}\Omega' \Delta J^2 + |H_\mathbf{m}| \cos(\phi) + \mathrm{const}, \qquad \Omega' = \left. \frac{\partial \Omega}{\partial J} \right|_{\Omega=0} \tag{5}$$

The nonlinear bounce frequency $\omega_{\mathrm{bN}}$ of periodic motion in $\phi$ is given by an elliptic integral. Its maximum value is estimated by linearization around $\phi = 0$ and is given by $\omega_{\mathrm{bN,max}} = |\Omega' H_\mathbf{m}|^{1/2}$.

**Collision operator in action-angle variables and nonlinear attenuation**

For computation of the non-axisymmetric part of the distribution function required for the torque density, only scattering by field particles across resonance zones is required, which makes it possible to represent $\hat{L}_C$ by

$$\hat{L}_C f \approx D_{\mathrm{res}} \frac{\partial^2 f}{\partial \Delta J^2}, \qquad D_{\mathrm{res}} = D^{vv} \left( \frac{1}{\Omega'} \frac{\partial \Omega}{\partial v} \right)^2 + D^{\chi\chi} \eta \left( \left\langle \frac{1}{B} \right\rangle_b - \eta \right) \left( \frac{1}{\Omega'} \frac{\partial \Omega}{\partial \eta} \right)^2 \tag{6}$$

The second expression for $D_{\mathrm{res}}$ includes collisional scattering coefficients $D^{vv}$ and $D^{\chi\chi}$ in velocity module $v$ and pitch angle $\chi$ and a bounce average $\langle \cdot \rangle_b$ of the inverse magnetic field module. Considering the steady-state case with a solution $f_0$ for the axisymmetric Hamiltonian $H_0$ and the perturbed distribution function $f = f_0 + f_1$ in Eq. (1), one obtains the evolution equation

$$\{f_1, H\} - \hat{L}_C f_1 = \{f_0, H_1\} = -\mathbf{m} \cdot \frac{\partial f_0}{\partial \mathbf{J}} |H_\mathbf{m}| \sin(\phi). \tag{7}$$

Rescaling to dimensionless variables in this equation by

$$x \equiv \Delta J \,\mathrm{sign}(\Omega') \left| \frac{\Omega'}{H_\mathbf{m}} \right|^{1/2}, \qquad D \equiv \frac{D_{\mathrm{res}} |\Omega'|^{1/2}}{|H_\mathbf{m}|^{3/2}}, \qquad g = f_1 \left| \frac{\Omega'}{H_\mathbf{m}} \right|^{1/2} \left( \mathbf{m} \cdot \frac{\partial f_0}{\partial \mathbf{J}} \right)^{-1}, \tag{8}$$

an equation depending only on one parameter $D$ results with

$$x \frac{\partial g}{\partial \phi} + \sin\phi \left( \frac{\partial g}{\partial x} + 1 \right) - D \frac{\partial^2 g}{\partial x^2} = 0. \tag{9}$$

The torque density (2) is proportional to thermodynamic forces $A_1, A_2$ as follows,

$$T_{\varphi}^{NA} = \frac{n_\alpha e_\alpha}{c}\frac{\mathrm{d}\psi_{\mathrm{pol}}}{\mathrm{d}r}(D_{11}A_1 + D_{12}A_2), \quad A_1 = \frac{1}{n_\alpha}\frac{\partial n_\alpha}{\partial r} + \frac{e_\alpha}{T_\alpha}\frac{\partial \Phi}{\partial r} - \frac{3}{2T_\alpha}\frac{\partial T_\alpha}{\partial r}, \; A_2 = \frac{1}{T_\alpha}\frac{\partial T_\alpha}{\partial r}. \tag{10}$$

The notation here follows the quasilinear version in [5, 6]. For each canonical **m**, diffusion coefficients $D_{11}$ and $D_{12}$ are given by an integral along the resonance in phase space parameterized by normalized velocity $u = v/v_\mathrm{T}$ and $\eta = v_\perp^2/(v^2 B)$,

$$\begin{pmatrix} D_{11} \\ D_{12} \end{pmatrix} = \frac{\pi^{3/2} n^2 c^2 v_T}{e_\alpha^2 S}\frac{\mathrm{d}r}{\mathrm{d}\psi_{\mathrm{pol}}}\int_0^\infty du\, u^3 e^{-u^2}\tau_b |H_\mathbf{m}|^2 \left| m_2 \frac{\partial \omega_b}{\partial \eta} + n\frac{\partial \Omega^3}{\partial \eta}\right|^{-1} \begin{pmatrix} 1 \\ u^2 \end{pmatrix} \Theta(D). \tag{11}$$

The result differs from the quasilinear case by the nonlinear attenuation parameter

$$\Theta = \Theta(D) = -\frac{1}{\pi^2}\int_{-\infty}^{\infty} \mathrm{d}x \int_{-\pi}^{\pi} \mathrm{d}\phi \sin(\phi) g(x,\phi). \tag{12}$$

**Numerical implementation, results and discussion**

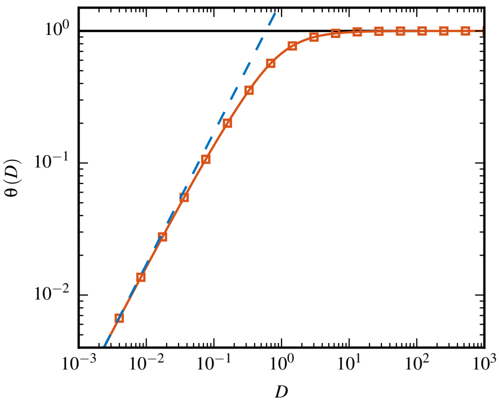


*Figure 1:* Attenuation factor $\Theta$ (□) depending on diffusion parameter $D$. Quasilinear (-) and nonlinear limit (- -) $\Theta \approx 1.7555267 \cdot D$.

Results including nonlinear attenuation were obtained for the circular tokamak used in [5, 6] using an extended version of the code NEO-RT that has been originally developed for the quasilinear case. The modifications include the nonlinear attenuation factor $\Theta(D)$ inside the integrals for the diffusion coefficients and the calculation of $D$ from plasma parameters. $\Theta(D)$ has been pre-tabulated from a numerical calculation for the solution of Eq. (12). Plasma parameters and perturbation amplitudes were chosen representative for a medium sized tokamak (deuterium plasma, density $n \approx 2 \cdot 10^{13}$ cm$^{-1}$, ion temperature $T_\mathrm{i} \approx 1.5$ keV) with RMPs. The figures show results for computations with a perturbed magnetic field module $B = B_0 \cdot (1 + \varepsilon_\mathrm{M}\cos(m\theta + n\varphi))$ by a single harmonic in Boozer angles $(\theta, \varphi)$ with $m = 0$ and $n = 3$, evaluated at a flux surface of aspect ratio $A = 10$.

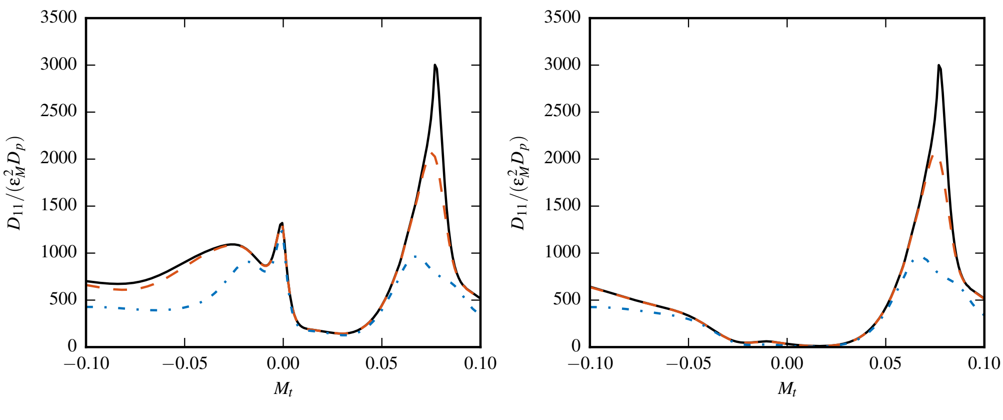


*Figure 2:* Normalized diffusion coefficient $D_{11}$ depending on Mach number $M_t$ at quasilinear limit (-) and for finite relative perturbation amplitudes $\varepsilon_\mathrm{M} = 10^{-3}$ (- -) and $10^{-2}$ (-·). Left: All resonances including superbanana resonance ($m_2 = 0$) with peak near $M_t = 0$. Right: Only drift-orbit resonances ($m_2 \neq 0$) with peak at $M_t = 0.07$.

Figure 1 shows the dependency of the nonlinear attenuation factor $\Theta(D)$ of the collisional parameter $D$ defined in Eq. (8). For the perturbation considered here, the latter scales approximately as $D \propto \nu_{\mathrm{eff}}\varepsilon_{\mathrm{M}}^{-3/2}$. In the nonlinear low-collisionality limit $D \ll 1$, the asymptotic dependency $\Theta(D)$ is linear. For $D \gg 1$, the quasilinear case $\Theta = 1$ without attenuation is reached.

In Figure 2, the the non-ambipolar radial diffusion coefficient $D_{11}$ normalized by the plateau coefficient $D_p = \pi q v_T^3/(16R\bar{\omega}_{c\alpha}^2)$ and the squared relative magnetic perturbation amplitude $\varepsilon_{\mathrm{M}}^2$ is plotted over the Mach number $M_t = \Omega_{tE}R/v_T$, which is proportional to the radial electric field $E_r$. Here, $v_{\mathrm{T}}$ is the thermal velocity and $R$ the major radius. To demonstrate the importance of nonlinear attenuation for drift-orbit resonance regimes, the superbanana resonance $m_2 = 0$, which is dominating around the electric zero with $M_t = 0$, has been excluded in the right plot. Nonlinear effects are clearly visible at $\varepsilon_{\mathrm{M}} = 10^{-3} - 10^{-2}$ and are more pronounced at higher perturbation amplitudes and higher absolute Mach number values.

Figure 3 shows the transition between quasilinear (superbanana plateau) and nonlinear (superbanana) resonant transport regimes at $M_t = -0.036$, where most contributions are caused by the superbanana resonance. Similar to the attenuation parameter, the normalized diffusion coefficient reaches the quasilinear limit at small $\varepsilon_{\mathrm{M}} < 10^{-3}$ and a nonlinear behavior at large $\varepsilon_{\mathrm{M}} > 10^{-3}$. As visible in Figure 2 at $M_t \approx 0.07$, for drift-orbit resonances the nonlinear onset is reached already at smaller perturbation amplitudes in this case.

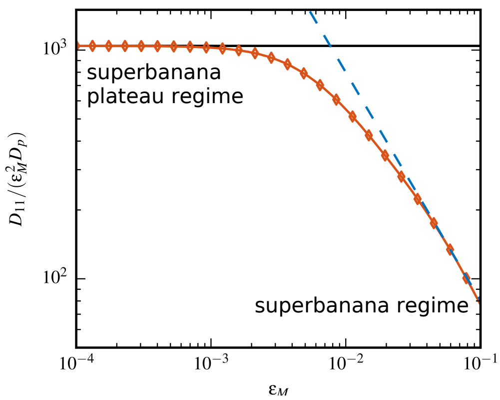


*Figure 3:* Normalized $D_{11}$ over perturbation strength at $M_t = -0.036$ (superbanana resonance). Quasilinear (-), nonlinear limit (- -).

## Conclusion and Outlook

Quasilinear calculations of NTV torque provide an upper limit for contributions from resonant transport regimes. In this work, the quasilinear Hamiltonian formalism underlying the code NEO-RT has been extended to include nonlinear attenuation effects, enabling the prediction of NTV torque in transition regimes between quasilinear and nonlinear limit. At perturbation amplitudes from RMPs of a few tenths of a percent, analysis of nonlinear attenuation is necessary in order not to overestimate toroidal torque in resonant transport regimes. The presented results show that at parameters for typical medium sized tokamaks, the overall attenuation can be in the order of tens of percents, depending on Mach number and perturbation strength. To reinforce these claims, benchmarks with numerical codes and comparison to experimental data are desirable.

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