Non-monotonic features in the runaway electron tail

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Radiation losses can significantly affect the momentum space dynamics of energetic runaway electrons. We find that synchrotron radiation reaction can lead to the appearance of a non-monotonic feature – a "bump" – in the runaway tail. It can be a potential source for kinetic instabilities limiting the formation of runaway beams. We derive threshold conditions for the bump to appear and to be physically relevant, and provide a lower bound on its location in momentum space. Numerical calculations with CODE [1] support the analytical results.

Kinetic equation Radiation reaction increases with perpendicular momentum. The range of perpendicular momenta where the accelerating field can overcome the friction plus the radiation reaction then becomes limited. Pitch-angle scattering of electrons out of this "runaway region" leads to an exponential decay of the electron distribution in the far-tail. For moderate (but relativistic) momenta, the return fluxes of electrons into the runaway region due to collisional friction and radiation reaction can overcome the outflow due to pitch-angle scattering, which can lead to the appearance of a local maximum in the distribution function. In the following we assume that such a non-monotonic feature of the steady state distribution exists and we study its properties.

The kinetic equation used here is derived from the guiding center kinetic equation for the gyro-angle averaged distribution function F, discussed in [2]. We consider a homogeneous plasma and a straight magnetic field line geometry, and assume the momentum range of interest to be much higher than the thermal momentum of electrons, which leads to

$$\frac{\partial F}{\partial \tau}+\frac{1}{s^2}\frac{\partial}{\partial s}\left[s^2\left(\hat{E}\xi-\sigma\gamma s(1-\xi^2)-\frac{\gamma^2}{s^2}\right)F\right]+\frac{\partial}{\partial \xi}\left[(1-\xi^2)\left(\hat{E}\frac{F}{s}+\frac{\sigma\xi}{\gamma}F-\frac{\gamma}{s}\frac{1+Z_{\mathrm{eff}}}{2s^2}\frac{\partial F}{\partial \xi}\right)\right]=0,$$

where $s=p/(m_e c)$ is the dimensionless momentum, $\xi=s_{\|}/s$ is the pitch-angle cosine, $\tau=eE_c t/(m_e c)$ is the normalized time with $E_c=n_e e^3\ln\Lambda/(4\pi\varepsilon_0^2 m_e c^2)$ the critical field for runaway generation, n_e the electron density, m_e the electron rest mass, c the speed of light, ϵ_0 the vacuum permittivity, e the elementary charge, and $\ln\Lambda$ the Coulomb logarithm. Furthermore $\hat{E}=-E_{\|}/E_c$ is the normalized parallel electric field, $\gamma=(1+s^2)^{1/2}$ is the relativistic factor, and Z_{eff} is the effective ion charge, $\sigma=2\Omega_e^2/(3\ln\Lambda\,\omega_{pe}^2)$ quantifies the strength of the radiation reaction (relative to collisional friction), where $\Omega_e=eB/m_e$ and $\omega_{pe}^2=e^2n_e/(\epsilon_0 m_e)$ are the gyro-, and plasma frequencies.

It is convenient to work in parallel and perpendicular momenta, $s_\parallel$ and $s_\perp$, since we will focus on the distribution function for small $s_\perp$. At an extremum of the distribution function located along the $s_\parallel$ axis (satisfying $(\partial_{s_\parallel} F)|_{(s_\parallel,0)} = 0$) the kinetic equation simplifies to

$$L(s_\parallel) \equiv 2\left(\sigma s_\parallel + \sqrt{1+s_\parallel^2}\right) - (1+s_\parallel^2)\sigma\kappa/\bar{E} = 0 \tag{1}$$

where $\bar{E} = (\hat{E}-1)/[2(1+Z_{\text{eff}})]$ and $\kappa(s_\parallel) = -(\hat{E}-1)/(2\sigma F)(\partial_{s_\perp s_\perp} F)|_{s_\perp=0}$. The order unity parameter κ is assumed to be slowly varying with $s_\parallel$, and it is defined so that it approaches 1 asymptotically in the far tail, where the distribution should approach Eq. (12) of Ref. [3].

Thresholds and bump location To find the threshold for the bump formation we use that the distribution should have an inflection point when the bump is about to appear. In that case $L(s_\parallel)$ of Eq. (1) and its derivative L' vanish simultaneously. It can be shown that the inflection point will not appear at high $s_\parallel$. Thus we may use $(1+s_\parallel^2)^{1/2} \approx 1 + s_\parallel^2/2$ to obtain simpler expressions (for more details consult [2]). Neglecting κ' corrections we find that the bump in the distribution appears if $\bar{E} > \kappa \sim 1$ or σ is below the threshold given by

$$\sigma_0 = \frac{3\kappa/\bar{E} + \sqrt{8+\kappa^2/\bar{E}^2}}{2\left(\kappa^2/\bar{E}^2 - 1\right)}. \tag{2}$$

Right at the threshold the inflection point is located at $s_{\parallel 0} = \sigma[(1+\sqrt{1+4\sigma^2})/2]^{-1} < 1$.

We can find an approximate expression for the bump location when it appears at high $s_\parallel$, in which case we may use $(1+s_\parallel^2)^{1/2} \approx s_\parallel$. In this limit, by assuming the characteristic perpendicular width of the distribution ($\propto \kappa^{-1/2}$) to be approximately constant, and looking for separable solutions of the form $F = h(s_\parallel)g(s_\perp)$, we find from the kinetic equation that

$$(\hat{E}-1)s_\parallel h' - 2(1+\sigma)h = -(1+Z_{\text{eff}})W^2 s_\parallel h, \tag{3}$$

with $W^2 = 2\sigma\kappa/(\hat{E}-1)$, for which $h(s_\parallel) \sim s_\parallel^{2(1+\sigma)/(\hat{E}-1)} \exp\left[-W^2(1+Z_{\text{eff}})/(\hat{E}-1)s_\parallel\right]$ is a solution. The function $h(s_\parallel)$ has a maximum at

$$s_\parallel = \frac{2\bar{E}}{\kappa}\frac{1+\sigma}{\sigma}. \tag{4}$$

Numerical investigation suggests that the order unity parameter κ is usually smaller than 1 at the bump. Thus Eq. (4) for $\kappa = 1$ is a lower bound for the bump location in momentum space.

For small values of σ, the bump would appear at high parallel momenta. Defining some upper limit of physical interest, $s_{\parallel,\text{L}}$, Eq. (4) may be used to find an estimate for a lower "practical limit" in σ for the appearance of the bump. Namely, if σ is smaller than

$$\sigma_{\text{L}} = \left[(s_{\parallel,\text{L}}\kappa)/(2\bar{E}) - 1\right]^{-1}, \tag{5}$$

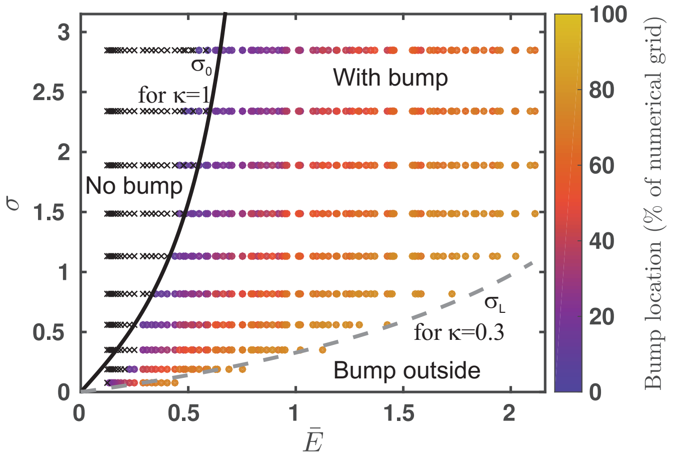

Figure 1: *CODE simulations yielding steady state solutions with (circles) or without (crosses) a bump in the runaway tail. Bumps above $s_\parallel = 27$ are excluded ("Bump outside"). Solid line: bump formation threshold Eq. (2) for $\kappa = 1$. Dashed line: practical threshold Eq. (5) for $s_{\parallel,\mathrm{L}} = 27$ and $\kappa = 0.3$. The color coding shows the location of the bump relative to $s_\parallel = 34$.*

for $\kappa = 1$, then a bump would only appear at some large parallel momentum above $s_{\parallel,\mathrm{L}}$, which is then deemed physically irrelevant. Note that if the bump is in the far tail, κ can be significantly less than unity, as will be shown in the next section using numerical simulations. Letting $\kappa < 1$ increases the practical limit in σ. For a normalized electric field higher than $\bar{E} = s_{\parallel,\mathrm{L}}/2$, the bump *always* appears above $s_{\parallel,\mathrm{L}}$ for any value of σ.

Numerical results The numerical results were obtained using the continuum simulation tool CODE (COllisional Distribution of Electrons), used in its time-independent mode. CODE solves the two dimensional momentum space kinetic equation in a homogeneous plasma, using a linearized Fokker-Planck operator valid for arbitrary energies, and including the radiation reaction force [4].

To investigate the validity of our analytical calculations, we performed a scan in the parameter space with CODE. The electron temperature and density were held constant at $T_e = 1\,\mathrm{keV}$ and $n_e = 5 \cdot 10^{18}\,\mathrm{m}^{-3}$, while the magnetic field, the electric field, and the effective ion charge were varied over the ranges $B \in [1,6]\,\mathrm{T}$, $\hat{E} \in [2,14]$ and $Z_{\mathrm{eff}} \in [1,3]$. The simulations used 950 momentum grid points, 130 Legendre modes in ξ, and a highest resolved momentum of $s = 34$.

The results of the scan are presented in Fig. 1, where circle (cross) symbols correspond to distributions with (without) a bump. The color coding reflects the location of the bump, with 100% in the color bar corresponding to $s_\parallel = 34$. Simulations with a bump above $s_\parallel = 27$ are excluded from the figure. Increasing the normalized electric field $\bar{E}$ or decreasing the radiation reaction strength σ moves the bump towards larger momenta, as expected from Eq. (4).

Good agreement is found between the numerical calculations and the bump formation threshold σ_0 (solid curve), Eq. (2) for $\kappa = 1$. The "no bump" solutions mostly obey the analytical threshold and fall to the left of the threshold curve. There are some solutions with bump in this region as well, since κ at the bump can be less than unity, which would move the threshold to the left. While the qualitative behavior of the threshold is still captured, the expression for the threshold begins to fail quantitatively for $\sigma < 0.5$, showing that the slowly varying κ approximation breaks down. The lower right corner of the plot is not populated, because the simulations where the bump would have appeared above $s_{\parallel} > 27$ were excluded, since they would be affected by the proximity to the end of the momentum grid. With $s_{\parallel,\mathrm{L}} = 27$, $\kappa = 0.3$ is needed for the practical threshold, σ_L of Eq. (5) (dashed line), to correspond well to the boundary of the region of excluded points. Thus, κ can be significantly lower than unity for a bump at large momentum.

Conclusions The time-asymptotic steady-state runaway distribution can become non-monotonic due to synchrotron radiation reaction. This non-monotonic feature presents a potential source for bump-on-tail instabilities, which can play a role in limiting the formation of large runaway beams. A threshold condition for the appearance of the bump and a lower limit for its location in momentum space were derived. Our analytical results show good agreement with numerical simulations obtained using the CODE solver.

For a normalized electric field $\bar{E} > 1$, the steady state electron distribution always exhibits a bump, independently of the strength of the radiation reaction (for $\sigma > 0$). For $\bar{E} < 1$, the appearance of the bump is well correlated with the σ threshold, Eq. (2). However, if the bump would appear at a very high parallel momentum it can take too long time for it to develop, compared to other time scales of the physical system. (The temporal evolution of the distribution function is discussed in [5].) If σ is lower than a "practical" threshold, Eq. (5), the bump will appear at a momentum above some pre-specified limit, $s_{\parallel,\mathrm{L}}$. For $\bar{E} > s_{\parallel,\mathrm{L}}/2$, this is satisfied for any σ.

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