Finite-aspect-ratio effects on neoclassical transport coefficients, revisited

M. Taguchi

College of Industrial Technology, Nihon University, Narashino, 275-8576, Japan

The neoclassical moment method is improved by increasing the accuracy of approximation to the linearized Fokker-Planck collision operator. In this paper, we apply this improved method to the calculation of ion flow velocity for a plasma with one impurity species.

Let us write the perturbed distribution function as $f_{a1} = -(I_{\parallel}/\Omega_a)f_{a0} + g_a$ in an axisymmetric magnetic field $B = I(\psi)\nabla\phi + \nabla\psi \times \nabla\phi$, where $v_{\parallel} = B \cdot v/B$, $\Omega_a$ is the Larmor frequency, $2\pi\psi$ is the poloidal flux and $\phi$ is the toroidal angle. Then, in the banana regime, the function $g_a$ for trapped particles ($\lambda > \lambda_c$) is identically zero, where $\lambda = (1 - v_{\parallel}^2/v^2)/B$ and $\lambda_c = 1/B_{\text{max}}$.

The analytic function $g_a$ for passing particles ($\lambda < \lambda_c$) can be obtained from the solubility condition by using some sort of approximation to the linearized Fokker-Planck collision operator $C_{ab}(f_{a1}, f_{b1})$.

We introduce the following approximate collision operator:

$$
C_{ab}(f_{a1}, f_{b1}) \simeq v_D^{ab}(v)\mathcal{L}(f_{a1}) + \sum_{l=0}^{3} P_l(\xi)f_{a1}^l C_{ab}(f_{a1}, f_{b1}^l)
$$

with

$$
\hat{C}_{ab}(f_{a1}, f_{b1}^l) = C_{ab}(f_{a1}^l, f_{b1}^l) + \frac{l(l+1)}{2} v_D^{ab}(v) f_{a1}^l,
$$

where $f_{a1}^l = (l+1/2) \int_{-1}^{1} P_l(\xi) f_{a1} d\xi$, $C_{ab}(P_l(\xi) f_{a1}^l(v), P_l(\xi) f_{b1}^l(v)) = P_l(\xi) C_{ab}(f_{a1}^l(v), f_{b1}^l(v))$, $\xi = v_{\parallel}/v$, $\mathcal{L}$ is the pitch-angle scattering operator, the deflection collision frequency $v_D^{ab}(v) = v_{ab}[\text{erf}(v/v_b) - G(v/v_b)](v_a/v)^3$ with $G(x) = [\text{erf}(x) - (2x/\sqrt{\pi})\exp(-x^2)]/(2x^2)$ and $v_{ab} = 4\pi n_b e_a e_b^2 \lambda/(m_a^2 v_a^3)$, thermal velocity $v_a = \sqrt{2T_a/m_a}$, and $n_a$ and $T_a$ are the number density and temperature, and $m_a$ and $e_a$ are the mass and charge. Using this approximate collision operator, we can obtain the distribution function for passing particles in the form

$$
g_a = \frac{1}{2} \frac{\sigma}{v_D^{ab}(v)} \int_{\lambda_c}^{\lambda_c} \frac{d\lambda}{\sqrt{1 - \lambda B}} G_{a1}(v) + \frac{5}{8} \frac{\sigma}{v_D^{ab}(v)} \int_{\lambda_c}^{\lambda_c} \frac{\lambda d\lambda}{\sqrt{1 - \lambda B}} G_{a2}(v),
$$

where $\sigma = v_{\parallel}/|v_{\parallel}|$, $v_D^{ab}(v) = \sum_b v_D^{ab}(v)$, $\langle \cdot \rangle$ denotes the flux-surface average,

$$
G_{a1} = \sum_b \left[ \hat{C}_{ab}^{1}(K_{a1}^1, K_{b1}) + \hat{C}_{ab}^{3}(K_{a3}, K_{b3}) - C_{ab} \left( \frac{IB}{\Omega_a} f_{a0}^f, \frac{IB}{\Omega_b} f_{b0}^f \right) \right]
$$

and

$$
G_{a2} = \sum_b \left[ \frac{7}{3} F_1 \hat{C}_{ab}^{3}(K_{a1}, K_{b1}) - \hat{C}_{ab}^{3}(K_{a3}, K_{b3}) \right]
$$
with \( F_t = 1 - \langle B^3 \rangle / \langle B^2 \rangle \langle B^4 \rangle \). The function \( K_{a1}(v) \equiv \langle B_{g_{a1}} \rangle \) is determined by the equation

\[
\frac{f_t}{f_c} v_{c}^{d} (v) K_{a1} - \sum_{b} C_{a1}^{1} (K_{a1}, K_{b1}) - \Delta t \sum_{b} C_{a1}^{3} (K_{a1}, K_{b1}) = - \sum_{b} C_{a1}^{1} \left( \frac{1B}{\Omega_{a} v f_{b0}^{0}}, \frac{1B}{\Omega_{b} v f_{b0}^{0}} \right)
\]  

(6)

with

\[
\Delta t = \frac{7}{3} \left( f_t f_c + \bar{f}_t f_c \right),
\]

(7)

where \( f_t = 1 - f_c, \bar{f}_t = 1 - \bar{f}_c, \bar{f}_t = \bar{f}_t + f_t \bar{f}_c \).

The flow velocity is expressed in the form

\[
u_a = u_{a0} B - \frac{T_a}{m_a \Omega_a} \left( \frac{p_a}{p_0} + \frac{e_a \Phi^{\prime}}{T_a} \right) R^2 \nabla \varphi,
\]

(11)

where the poloidal flow \( u_{a0} \) is written in terms of the function \( K_{a1}(v) \) as \( u_{a0} = (4\pi/3n_a) \times (1/\langle B^2 \rangle) \int_0^\infty dv v^3 K_{a1} \).

We next explicitly calculate the poloidal flows of primary ions and impurities. Let us expand the function \( K_{a1}(v) \) in a series of the associate Laguerre polynomials of order 3/2 and retain only the first and second terms: \( K_{a1}(v) \approx (m_a/T_a) \langle B^2 \rangle v [u_{a0} - (2/5) q_{a0}/p_a (5/2 - v^2/v_a^2)] f_{a0} \), where \( p_a = n_a T_a \) and \( f_{a0} \) is the Maxwell distribution function. Inserting this expansion for \( K_{a1} \) into Eq.(6) and taking velocity moment with respect to \( v^3 \) and \( v^3 (5/2 - v^2/v_a^2) \) lead to a set of coupled algebraic equations for \( u_{a0} \) and \( q_{a0} \). From here, the subscripts \( i \) and \( I \) represent the primary and impurity ions. Assuming that the primary ions are in the banana regime and using the smallness of the mass ratio \( m_i/m_I \), we solve this coupled equations to find the poloidal flows \( u_{i0} \) and \( u_{I0} \):

\[
\begin{bmatrix}
  u_{i0} \\
  u_{I0}
\end{bmatrix}
= \frac{Ic T_i}{e_i \langle B^2 \rangle} \sum_{k=1}^{4} \begin{bmatrix} u_{ik} \\
  u_{Ik}
\end{bmatrix} A_k,
\]

(12)

where

\[
\begin{bmatrix}
  u_{i1} \\
  u_{i2} \\
  u_{i3} \\
  u_{i4}
\end{bmatrix}
= \frac{1}{D_i} \begin{bmatrix}
  \beta_1 s_1 \\
  \bar{\mu}_i s_2 - (3/2) \bar{\mu}_i s_1 \\
  \beta_1 s_3 \\
  -5\alpha \mu_2 \delta \delta T \beta_1 / D_i
\end{bmatrix},
\]

(13)
\[
\begin{bmatrix}
u_{I1} \\
u_{I2} \\
u_{I3} \\
u_{I4}
\end{bmatrix} = \frac{\delta}{D_iD_I} \left( \mu_{I3} + \sqrt{2 + \frac{15}{2} \delta} \right) \begin{bmatrix}
\beta_2 \\
\beta_3 \\
\beta_4 \\
\beta_5
\end{bmatrix} - \frac{\mu_{I2}}{D_I} \begin{bmatrix}
0 \\
0 \\
\sqrt{2 + 15 \delta/2} \\
5 \delta \delta_T
\end{bmatrix},
\]

(14)

and the thermal forces are defined by \( A_1 = p_i'/(p_i - (e_i/e_1)(T_i/T_r)(p_i'/p_I)), \ A_2 = T_i'/T_r, \ A_3 = (e_i/e_1)(T_i'/T_i), \ A_4 = (T_i/T_r)(e_i/e_1)(p_i'/p_I + e_i \Phi')/T_i \). The parameters in (13) and (14) are defined as \( \alpha = n_i e_i^2/(n_i e_i^2), \ \delta = (1/\alpha) \sqrt{m_i/m_i/(T_i/T_i)^{3/2}}, \ \delta_T = 1 - T_i/T_r, \ \beta_1 = \bar{\mu}_{i3} - s_2 + (3/2)(\bar{\mu}_{i2} - 3s_1/2), \ \beta_2 = (9s_1/4 + s_2)\bar{\mu}_{i1} + \bar{\mu}_{i2} - \bar{\mu}_{i1}\bar{\mu}_{i3}, \ \beta_3 = (3/2)(\bar{\mu}_{i1}\bar{\mu}_{i3} - \bar{\mu}_{i2}^2) + (9s_1/4 + s_2)\bar{\mu}_{i2}, \ \beta_4 = 9\bar{\mu}_{i1}/4 + 3\bar{\mu}_{i2} + \bar{\mu}_{i3} - 9s_1/4 - s_2, \ \beta_5 = -5(\alpha/D_i)\mu_{I2} \delta \delta_T \beta_4.

\[
D_i = (\bar{\mu}_{i1} + s_1)(\bar{\mu}_{i3} - s_2) - \left( \bar{\mu}_{i2} - \frac{3}{2} s_1 \right)^2,
\]

(15)

\[
D_I = (\mu_{I1} + \delta) \left( \mu_{I3} + \sqrt{2 + \frac{15}{2} \delta} \right) - \mu_{I2} (\mu_{I2} + 5 \delta \delta_T \delta),
\]

(16)

\[
s_1 = \frac{\alpha}{D_i} \left\{ \mu_{I1} \left( \mu_{I3} + \sqrt{2 + \frac{15}{2} \delta} \right) - \mu_{I2} (\mu_{I2} + 5 \delta \delta_T \delta) \right\},
\]

(17)

\[
s_2 = \frac{1}{D_i} \left\{ - (\mu_{I3} + \sqrt{2 + \frac{15}{2} \delta}) \left[ \sqrt{2}(\mu_{I1} + \delta) + \frac{13}{4} \alpha \mu_{I1} + \alpha \delta \right] \\
+ \left( \sqrt{2} + \frac{13}{4} \alpha \right) \mu_{I2} (\mu_{I2} + 5 \delta \delta_T \delta) \right\},
\]

(18)

\[
s_3 = -\frac{\alpha}{D_i} \mu_{I2} \left( \sqrt{2 + \frac{15}{2} \delta} \right).
\]

(19)

The viscosity coefficients for primary ions are written as

\[
\begin{bmatrix}
\bar{\mu}_{i1} \\
\bar{\mu}_{i2} \\
\bar{\mu}_{i3}
\end{bmatrix} = \begin{bmatrix}
\mu_{I1} \\
\mu_{I2} \\
\mu_{I3}
\end{bmatrix} - \Delta_t \begin{bmatrix}
(\hat{C}_{ii}^3)_{00} \\
(\hat{C}_{ii}^3)_{01} \\
(\hat{C}_{ii}^3)_{11}
\end{bmatrix},
\]

(20)

where the conventional viscosity coefficients are given by \( \mu_{I1} = (f_i/f_c)[\sqrt{2} + \alpha - \log(1 + \sqrt{2})] \), \( \mu_{I2} = (f_i/f_c)[-2\sqrt{2} - 3\alpha/2 + (5/2)\log(1 + \sqrt{2})] \) and \( \mu_{I3} = (f_i/f_c)[(39/8)\sqrt{2} + 13\alpha/4 - (25/4)\log(1 + \sqrt{2})] \), and the matrix elements of \( \hat{C}_{ii}^3 \) are calculated as follows: \( (\hat{C}_{ii}^3)_{00} = -((1087/63)\sqrt{2} + (589/21)) \log(1 + \sqrt{2}) \), \( (\hat{C}_{ii}^3)_{01} = -(143/126)\sqrt{2} + (55/21)\log(1 + \sqrt{2}) \), \( (\hat{C}_{ii}^3)_{11} = (50923/504)\sqrt{2} - (13625/84)\log(1 + \sqrt{2}) \). The viscosity coefficients for impurities in the plateau to Pfirsch-Schlüter regime are given by

\[
\begin{bmatrix}
\mu_{I1} \\
\mu_{I2} \\
\mu_{I3}
\end{bmatrix} = \frac{\tau_{II}}{n_i} \frac{8\pi}{3} \int_0^\infty dv \frac{v^4}{v_f^2} \nu_f^2(v) \left( \frac{f_{II}^*}{1 + f_{II}^*/f_{II}} \right) f_{I0} \left[ \frac{1}{v_f^2} - \frac{5}{2} \left( \frac{v_f^2}{v_f^2} \right)^2 \right],
\]

(21)
where \( \tau_{II} = \frac{3\sqrt{\pi}}{4\nu_{II}} \), 

\[
\tilde{f}_{II} = \frac{3}{5} \langle (b \cdot \nabla B)^2 \rangle \frac{v^2}{v_D^2(v) v_f^2(v)}, \quad f_{\nu}^* = \frac{3\pi^2 v}{16 \varepsilon^2} \frac{1}{R_q v_D(v)},
\]

\( v_f^2(v) = v_{II} \left\{ [\text{erf}(v/v_I) - 3G(v/v_I)](v_I/v)^3 + (8/v_I)G(v/v_I) + (8/3\sqrt{\pi})\delta \right\} \), 

\( v_D^2(v) = v_{II} \times \left\{ [\text{erf}(v/v_I) - G(v/v_I)](v_I/v)^3 + (4/3\sqrt{\pi})(T_i/T_I)\delta (v_I/v)^2 \right\} \) and \( f_{\nu}^* \) is obtained for a model magnetic field with circular flux surfaces, i.e., \( B = B_0/(1 + \varepsilon \cos \theta) \). The impurity viscosity coefficients in the banana regime are obtained in the form:

\[
\begin{bmatrix}
\mu_{I1} \\
\mu_{I2} \\
\mu_{I3}
\end{bmatrix} = \begin{bmatrix}
\tilde{\mu}_{I1}(\alpha = 0) \\
\tilde{\mu}_{I2}(\alpha = 0) \\
\tilde{\mu}_{I3}(\alpha = 0)
\end{bmatrix} + \frac{f_i}{f_e} \frac{T_i}{T_I} \delta \begin{bmatrix}
2/3 \\
-2/3 \\
5/3
\end{bmatrix}.
\]

Finally we show the normalized poloidal flows of primary ions due to the thermal force \( A_2 \) in the model magnetic field with circular flux surfaces. The normalized flows \( u_{I2} \) for \( \alpha = 0, 1 \) and \( 4 \) are plotted as a function of the inverse aspect ratio \( \varepsilon \) in Fig. 1. The impurity ions are assumed to be (a) in the plateau to Pfirsch-Schlüter regime and (b) in the banana regime. The normalized flows obtained by the conventional moment method are larger than those by our method by up to about 20% in the range of intermediate aspect ratio. We also plot those conventional flows by dotted curves in Fig.1.

![Figure 1: Normalized poloidal flow \( u_{I2} \) versus inverse aspect ratio \( \varepsilon \). For comparison, the flows obtained by the conventional moment method are also plotted by the dotted curves. The parameter \( \nu_{*I} \) in (a) is defined by \( \nu_{*I} = (16/3\pi)(f_i/f_e)(1 - \varepsilon^2)R_q/(\varepsilon^2v_I\tau_{II}) \).](image)

References
