Global structure and magnetic components of geodesic acoustic modes in shaped tokamak plasmas

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Introduction

Winsor et al. [1] first derived the geodesic acoustic mode (GAM) to be an electrostatic perturbation, localized within a narrow radial region of the plasma. However, recent experimental evidence that the GAM also has a magnetic component with poloidal mode number \(m = 2\), existing outside the GAM surface, and even outside the plasma. Such a signal in \(\delta B_\theta\), with angular dependence \(\sin 2\theta\), has been detected in JT-60U [2] and in TCV [3] (possibly in other machines too), and was predicted theoretically in [4] in a study of the global structure of the corresponding eigenmode within ideal MHD. The ideal MHD model is unable to resolve the detailed fine structure of the GAM, but unlike standard kinetic treatments, it does naturally include the combination of the effects of compressibility, full electromagnetic effects, and mode coupling due to toroidal or shaping geometry effects, which are required to show that the mode has a global structure [5], including a surrounding “magnetic halo” existing also outside the plasma. While the theory in [4] was developed for toroidally rotating plasmas with circular cross section, the present work extends the analysis in those papers to (static) plasmas with a non-circular cross section. Such an extension of the analysis shows that i) the major effect of plasma shaping on the GAM frequency within the ideal MHD model is a decreasing GAM frequency with elongation, and ii) a non-circular plasma cross section, as well as a finite aspect ratio, induces other Fourier components than \(m = 2\) in the “magnetic halo” surrounding the GAM surface.

Equilibrium

We consider an axisymmetric, toroidal plasma equilibrium with magnetic field 

\[ B = \nabla \phi \times \nabla \psi + F(\psi) \nabla \theta = B^{\theta} e_{\theta} + B^{\phi} e_{\phi}, \]

where \(B^{\theta} = F / R^2\), \(B^{\phi} = \mu B^{\theta}\) (\(\mu \equiv 1/q\)), and \(e_{\theta}\) and \(e_{\phi}\) are tangent-basis vectors of the flux coordinates \((r, \theta, \varphi)\). We define these coordinates, as well as the plasma geometry, by the transformation

\[
\begin{align}
\rho(r, \theta) &= r + \varepsilon \sum_{m=1,2,3} \Delta_m^{(1)}(r) \cos m\theta + \varepsilon^2 \sum_{m=0,1,2,3,4,5,6} \Delta_m^{(2)}(r) \cos m\theta + \cdots \\
\omega(r, \theta) &= \theta + \varepsilon \sum_{m=1,2,3} \tau_m^{(1)}(r) \sin m\theta + \varepsilon^2 \sum_{m=1,2,3,4,5,6} \tau_m^{(2)}(r) \sin m\theta + \cdots
\end{align}
\]

(1a)

(1b)

Here, \((\rho, \omega, \varphi)\) is a polar-toroidal system related to the cylindrical system \((R, \phi, Z)\) by the transformations \(R = R_0 + \rho \cos \omega, Z = \rho \sin \omega\) and \(\phi = -\varphi\). The quantity \(R_0\) denotes the major radius of the plasma center in the limit \(\varepsilon \rightarrow 0\), where \(\varepsilon\) is the inverse aspect ratio. Although the quantities \(\Delta_m^{(1)}(\propto \text{ellipticity})\) and \(\Delta_m^{(2)}(\propto \text{triangularity})\) are not directly related to the toroidicity of the plasma (they are assumed to be produced by external coils, or by a
non-circular wall), they are here assumed to be of the same order of magnitude as the Shafranov shift $\Delta_1^{(1)}$, and hence formally included in the expansion in Eq. (1) in the same way as $\Delta_1^{(1)}$. The quantities $r_m^{(p)}$ are chosen such that $(r, \theta, \varphi)$ becomes a straight field line system, accomplished by making $J/R^2$ a flux function, where $J = [\nabla r \cdot (\nabla \theta \times \nabla \varphi)]^{-1}$ is the Jacobian. The functions $\Delta_m^{(p)}(r)$ are thereafter calculated from the equilibrium equation $\nabla p - J \times B = 0$, where $p \sim \varepsilon^2 B_0^2$ is the pressure and $B_0$ the toroidal magnetic field.

Expansion of the dynamical MHD equations We represent the plasma displacement $\xi$ and the perturbed magnetic field $Q$ by their contravariant components, i.e. $\xi = \xi^r e_r + \xi^\theta e_\theta + \xi^\varphi e_\varphi$ and $Q = Q^r e_r + Q^\theta e_\theta + Q^\varphi e_\varphi$, respectively, while the equation of motion is resolved into its covariant components:

$$\rho \omega^2 \xi - \nabla(\delta P) + (B \cdot \nabla)Q + (Q \cdot \nabla)B = Ke^r + Le^\theta + Me^\varphi = 0$$  \(2\)

Using the relations $e_i = g_{ij} e^j$ and $e_j \cdot (\partial e_i/\partial u^k) = [j, ik]$, where $g_{ij} = e_i \cdot e_j$ is the metric tensor and $[j, ik]$ the Christoffel symbol of the first kind, the quantities $K, L$ and $M$ in Eq. (2) can be expressed in relatively simple form in terms of the quantities $B^j, \xi^j, Q^j$ and $g_{ij}$. For axisymmetric modes, satisfying $\partial/\partial \varphi = 0$, we obtain for instance

$$L = \rho \omega^2 e_\theta \cdot \xi - \frac{\partial \delta P}{\partial \theta} - B^\varphi Q^\varphi \frac{\partial \delta \varphi}{\partial \varphi} + Q^r \frac{\partial}{\partial r} \left( B^\theta g_{\theta \varphi} \right) + B^\theta \frac{\partial Q^r}{\partial \theta} g_{r \varphi} + \frac{\partial}{\partial \theta} \left( B^\theta Q^\varphi g_{\theta \varphi} \right)$$  \(3\)

where $e_\theta \cdot \xi = g_{r \varphi} \xi^r + g_{\theta \varphi} \xi^\theta$ and $\delta P = -\xi^r \partial p/\partial r - \Gamma p (\nabla \cdot \xi) + B^\theta Q^r g_{r \varphi} + B^\theta Q^\varphi g_{\theta \varphi} + B^\varphi Q^\varphi g_{\varphi \varphi}$. Similar expressions are obtained for $K$ and $M$ in Eq. (2). Then, the contravariant $r$- and $\theta$-components of Faraday’s law $Q - \nabla \times (\xi \times B) = 0$ yields, for axisymmetric modes, $Q^r = B^\theta \partial \xi^r / \partial \theta$ and $Q^\theta = -J^{-1} \partial (J B^\theta \xi^r) / \partial r$, while the $\varphi$-component gives the equation $Q^\varphi + J^{-1} \partial (J B^\varphi \xi^r) / \partial r + B^\varphi (\partial \xi^\theta / \partial \theta - \mu \partial \xi^\varphi / \partial \varphi) = 0$. We then look for solutions to the system above in the form of a perturbation expansion

$$\xi^r = \varepsilon^2 \xi^r_{2(2)} \sin 2 \theta + \varepsilon^3 \sum_{m=1-5} \xi^r_{m(3)} \sin m \theta + \cdots$$  \(4a\)

$$\xi^\theta = \xi^\theta_{0(0)} + \varepsilon \xi^\theta_{1(1)} \cos \theta + \varepsilon^2 \sum_{m=0-4} \xi^\theta_{m(2)} \cos m \theta + \varepsilon^3 \sum_{m=0-5} \xi^\theta_{m(3)} \cos m \theta + \cdots$$  \(4b\)

$$\xi^\varphi = \varepsilon \xi^\varphi_{1(1)} \cos \theta + \varepsilon^2 \sum_{m=0-4} \xi^\varphi_{m(2)} \cos m \theta + \cdots$$  \(4c\)

This extends the expansion in [4] to the next order in $\varepsilon$ (for a static plasma). A similar extension of the expansion for $Q$ in [4] is also assumed, starting with $Q^r = \varepsilon^2 Q^r_{2(3)} \cos 2 \theta + \cdots$, $Q^\theta = \varepsilon^3 Q^\theta_{2(3)} \sin 2 \theta + \cdots$ and $Q^\varphi = \varepsilon^4 Q^\varphi_{1(4)} \sin \theta + \cdots$. Inserting these expansions into $K, L$ and $M$ in Eq. (2) leads to expansions of similar kind for these quantities, e.g.

$$L = \varepsilon^3 L_{1(3)}^4 \cos \theta + \varepsilon^4 \sum_{m=0-4} L_{m(4)}^4 \cos m \theta + \varepsilon^5 \sum_{m=0-5} L_{m(5)}^5 \cos m \theta + \cdots = 0$$  \(5\)
GAM solution To leading order, the equations above lead to the GAM solution in [4] (for a static plasma). For instance, \( L_0^{(4)} = 0 \) gives the GAM frequency \( \omega_{GAM}^2 = \omega_s^2 (2 + 1/q^2) \) as well as the eigenfunction \( \xi_0^{(0)}(r) = \xi \delta(r - r_0) \), while the equations \( L_2^{(4)} = 0 \) and \( K_2^{(4)} = 0 \) lead to an equation for the \( m = 2 \) perturbation \( \xi_2^{(2)}(r) \), from which the \( m = 2 \) component of the “magnetic halo” [5] surrounding the GAM surface can be calculated [4].

To the next order, two important extensions of the solution above are found. First, a correction to \( \omega_{GAM}^2 \) above is produced if \( \Delta_2^{(1)} \) in Eq. (1a) is finite. In terms of the elongation \( \kappa \cong 1 - 2 \Delta_2^{(1)}/r \), the corrected GAM frequency is, from \( L_0^{(5)} = 0 \), found to be given by

\[
\omega_{GAM}^2 \cong \omega_s^2 \left[ 2 + \frac{1}{q^2} - 2(\kappa - 1) - \frac{r}{2 \mu} \frac{d \kappa}{dr} \right]_{r = r_0}
\]  

As shown in Fig. 1, this leads to a substantial reduction of the GAM frequency in elongated plasmas, especially in the edge region of plasmas with strong magnetic shear, where the \( d \kappa/dr \propto \kappa S \) term becomes of importance.

This result is in line with and qualitatively similar to previous calculations of shaping effects on the frequency of the electrostatic GAM, e.g. in [6], and in qualitative agreement with experiments. We emphasize that it is only the ellipticity term \( \Delta_2^{(1)} \) in Eq. (1) that leads to a correction of the frequency, and neither triangularity \( \Delta_3^{(1)} \) nor squareness \( \Delta_4^{(1)} \) are found to have any effect on the GAM frequency (to this order). The ellipticity terms in Eq. (6) therefore constitute the major effect of plasma shaping on the GAM frequency predicted by the ideal MHD model.

Although all the shaping terms, except for \( \Delta_2^{(1)} \), turn out to have negligible effects on the GAM frequency, they are nonetheless found to modify the structure of the perturbed magnetic field surrounding the GAM surface. Especially, finite ellipticity and triangularity are found to induce Fourier components in \( Q \) with mode numbers \( m = 4 \) and \( m = 5 \), respectively, outside the GAM surface. In addition, ellipticity contributes to the \( m = 2 \) component while triangularity, together with the Shafranov shift, contributes to the \( m = 1 \) as well as the \( m = 3 \) components of \( Q \). As an example, the \( m = 5 \) component in the plasma can be found by solving the following differential equation for \( \xi_5^{(3)}(r) \):

\[
\left[ \frac{d}{dr} \left( \mu^2 r^3 \frac{d}{dr} \right) - 24 r \mu^2 \right] \xi_5^{(3)} + G_5 \left( \xi_2^{(3)}, \Delta_3^{(1)}, \mu \right) + \frac{d}{dr} \left( \frac{\omega_2^2}{\omega_A^2} r^3 V_{5a} \xi_0^{(0)} \right) + \frac{r^3}{\omega_A^2} \frac{d}{dr} \left( \frac{\omega_2^2}{\omega_A^2} V_{5b} \xi_0^{(0)} \right) - r^6 \frac{d}{dr} \left( \frac{4 \omega_2^2}{\omega_A^2} \frac{\mu R_0}{r^4} \xi_4^{(2)} \right) = 0
\]  

where \( G_5 \) accounts for a coupling to the \( m = 2 \) component, \( V_{5a,b} = V_{5a,b}(\Delta_3^{(1)}, \mu) \) and \( \xi_4^{(2)} \).
\( (16\mu r / 3R_0) \left( d\Delta_3^{(1)}/dr - 2\Delta_3^{(1)}/r \right) (15\mu^2 - 2)^{-1} \xi_0^{\theta(0)} \) is a toroidal side-band also induced by the triangularity. Notice the singularity at \( \mu^{-1} = q = \sqrt{7.5} \approx 2.7 \), which comes from resonance with sound waves propagating along the field lines. Equations similar to Eq. (7) are found also for the other components \( \xi_m^{r(3)}(r), m = 1 - 4 \), in Eq. (4a).

In Fig. 2 we show two examples of the Fourier components \( m = 1 - 5 \) that surround the GAM surface. By the delta-function character of \( \xi_0^\theta(0) \), discontinuities both in the amplitudes \( \xi_m^{r(3)} \) as well as in the derivatives \( d\xi_m^{r(3)}/dr \) are in general produced across the GAM surface. In both figures \( \kappa_a = 1.25, \delta_a = 0.25, \epsilon_a = 0.1 \) and \( \beta(r) = \beta(0)[1 - (r/a)^2]^2 \), where \( \beta(0) \) is chosen such that \( \beta_p(a) = 0.5 \). In Fig. 2a the amplitudes \( r\xi_m^{r(3)} \) are shown in a plasma with the \( q \)-profile \( q(r) = 1 + 3(r/a)^4 \) and GAM radius \( r_0 = 0.3a \), while in Fig. 2b, \( r_0 = 0.75a \), \( q(r) = 1 + 2(r/a)^6 \) and the Fourier components of \( rQ^\theta(0) = -rJ^{-1} \partial(JB^\theta \xi^r)/\partial r \) are plotted. Some similarities with the corresponding spectral components obtained in the numerical solution of the ideal MHD GAM by Berk et al. [5] can be seen in Fig. 2a. A typical feature seen in the kind of \( q \)-profile used in Fig. 2b, with small shear in the core region and large shear in the edge region, is small amplitudes inside the GAM radius (except \( m = 1 \)) and much stronger amplitudes outside the GAM radius. In both figures the boundary conditions \( \xi_m^{r(3)}(a) = 0 \) have been used for simplicity. The degree to which a “magnetic halo” will exist also outside the plasma if a vacuum region is included will be discussed in future work.

References