Neoclassical relationship between the radial electric field and the radial current in tokamak plasmas

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Introduction

One of the dominant sources of toroidal torque in current tokamak experiments is neutral beam injection (NBI) that can impart toroidal rotation in the injection direction through both the collisional slowing-down process and the charge separation of fast neutrals. The charge separation stemming from the radial deviation from the birth surface of fast ions produces the radial current of fast ions. To maintain quasi-neutrality, a plasma naturally produces the radial current flowing in the direction opposite to the fast-ion radial current, leading to a $\vec{j}\times\vec{B}$ torque [1, 2]. This phenomenon is due to the property of a large dielectric constant of a plasma [3] and the radial current is believed to be a polarization current. The polarization current is in proportion to the temporal variation of the radial electric field E_r and, thus, is capable of quickly compensating for the sudden fast-ion current due to the onset of NBI. Given a plasma with constant NBI in a steady state, the charge separation and the resultant $\vec{j}\times\vec{B}$ torque must persist as long as NBI is activated, but E_r has to be constant over time, implying that the polarization current no longer plays a role in compensating for the fast-ion radial current. In a steady state, the radial current must be driven by some mechanisms other than the polarization. To investigate the characteristics of a $\vec{j}\times\vec{B}$ torque stemming from the charge separation in both transient and steady-state phases, we analytically derive the equation in a rigorous manner, stipulating the neoclassical relationship between the radial electric field and radial current in tokamak plasmas, especially when heated by NBI.

The one-dimensional transport code TASK/TX [4] consisting of two-fluid equations coupled with Maxwell's equation has already produced the $\vec{j}\times\vec{B}$ torque in its system solely using the estimated source profiles of electrons and fast ions [5], with the aid of an orbit-following Monte Carlo code, OFMC [6]. This fact means that, unlike conventional transport codes, the basis equations of TASK/TX may be essentially capable of reproducing the characteristics derived by the above-mentioned equation. Accordingly, it is important to firmly establish an analytical framework that enables TASK/TX to successfully reproduce a $\vec{j}\times\vec{B}$ torque, by examining the basis equations of TASK/TX.

The relationship between E_r and the radial current

In axisymmetric flux-surface coordinates (ψ, θ, ϕ), the radial Ampère's law is given by

$$\varepsilon_0 \frac{\partial}{\partial t}\left(\langle|\nabla\psi|^2\rangle\frac{\partial\Phi}{\partial\psi}\right) = \langle\vec{j}^{\rm tot}\cdot\nabla\psi\rangle, \tag{1}$$

where the brackets denote the flux-surface average and $\vec{j}^{\rm tot}$ is the current summed over the species including fast ions, i.e. $\vec{j}^{\rm tot} \equiv \vec{j} + \vec{j}^{\rm fast}$. Other variables follow conventional notation.

Initially we consider the moment equation as follows:

$$m_s n_s \left.\frac{d\vec{u}_s}{dt}\right|_{\vec{x}} = -\nabla p_s - \nabla\cdot\overleftrightarrow{\pi}_s + e_s n_s(\vec{E} + \vec{u}_s\times\vec{B}) + \vec{R}_s, \tag{2}$$

where $\overleftrightarrow{\pi}_s$ denotes the viscosity tensor for species s; and $\vec{R}_s$, the exchange of momentum. Taking the toroidal projection, $R^2\nabla\phi$, of (2) and the flux surface-average gives

$$\langle\vec{j}_s\cdot\nabla\psi\rangle = -\langle R^2\nabla\phi\cdot(\vec{R}_s^{\rm C} + e_s n_s\vec{E})\rangle + m_s n_s\left\langle R^2\nabla\phi\cdot\frac{\partial\vec{u}_s}{\partial t}\right\rangle + \langle R^2\nabla\phi\cdot(\nabla\cdot\overleftrightarrow{\pi}_s - \vec{R}_s^{\rm nC})\rangle,$$

where the friction term has been decomposed into the first-order Coulomb friction force $\vec{R}_s^{\rm C}$ and the non-Coulomb friction force $\vec{R}_s^{\rm nC} \equiv \vec{R}_s - \vec{R}_s^{\rm C}$, which is usually small compared to $\vec{R}_s^{\rm C}$. To first order in transport ordering δ, the stress in a magnetized plasma may be approximated by the Chew-Goldberger-Low stress and this form of the stress conserves toroidal angular momentum in an axisymmetric system. The remaining stress may be due mainly to the turbulence. We sum $\langle\vec{j}_s\cdot\nabla\psi\rangle$ over the plasma species to obtain the radial current as follows:

$$\langle\vec{j}\cdot\nabla\psi\rangle = \langle\vec{j}^{\rm p}\cdot\nabla\psi\rangle + \sum_{s=e,i}\left[\langle R^2\nabla\phi\cdot\nabla\cdot\overleftrightarrow{\pi}_s{}^{(2)}\rangle - \langle R^2\nabla\phi\cdot\vec{R}_s^{\rm nC}\rangle\right], \tag{3}$$

where we have defined the polarization current as $\langle\vec{j}^{\rm p}\cdot\nabla\psi\rangle \equiv \sum_{s=e,i} m_s n_s \partial/\partial t\langle Ru_{s\phi}\rangle$.

Using the first-order incompressible flow within the flux surface, $\vec{u}_s = \omega_s R^2\nabla\phi + \hat{u}_{s\theta}\vec{B}$, where ω_s denotes the diamagnetic frequency and $\hat{u}_{s\theta}$, the contravariant component of the poloidal velocity, we have $\langle Ru_{s\phi}\rangle = (I/\langle B^2\rangle)\langle Bu_{s\|}\rangle + (\langle R^2\rangle/\langle B^2\rangle)\langle B_\theta^2\rangle(1+2\hat{q}^2)\omega_s$. Here, the factor $\hat{q}$ is defined by $\hat{q}^2 = I^2/(2\langle B_\theta^2\rangle)(\langle 1/R^2\rangle - 1/\langle R^2\rangle)$, identical to the safety factor q in the limit of the large aspect ratio. Because the pressure will be nearly constant relative to the decay time of poloidal velocity, $\tau_{\rm p} \sim \varepsilon\tau_{\rm ii}$, the polarization current can be written as the combination of the temporal variation of the parallel flow and E_r as follows:

$$\langle\vec{j}^{\rm p}\cdot\nabla\psi\rangle = \sum_{s=e,i} m_s n_s\left[\frac{I}{\langle B^2\rangle}\frac{\partial}{\partial t}\langle Bu_{s\|}\rangle - \frac{\langle R^2\rangle}{\langle B^2\rangle}\langle B_\theta^2\rangle(1+2\hat{q}^2)\frac{\partial}{\partial t}\frac{\partial\Phi}{\partial\psi}\right]. \tag{4}$$

Substituting the parallel momentum equation into (4) yields

$$\langle \vec{j}^{\mathrm{p}} \cdot \nabla\psi \rangle = \frac{I}{\langle B^2 \rangle} \sum_{s=e,i} \left[-\langle \vec{B} \cdot \nabla \cdot \overleftrightarrow{\Pi}_s \rangle + \langle \vec{B} \cdot \vec{R}_s^{\mathrm{nC}} \rangle \right] - \frac{1}{\mu_0 v_{\mathrm{A}}^2} \langle R^2 \rangle \langle B_\theta^2 \rangle (1 + 2\hat{q}^2) \frac{\partial}{\partial t} \frac{\partial \Phi}{\partial \psi}, \quad (5)$$

where $\overleftrightarrow{\Pi}$ represents the neoclassical viscous stress. This is the final form of the polarization current. The characteristic time scales on which the density, temperature, flow and poloidal magnetic field vary are in general much longer than those of E_r, when E_r varies in response to the non-ambipolar radial current. Only when considering the time scale much shorter than those characteristic time scales, can we neglect the time change in the parallel flow in (4) [7]. We therefore see that the polarization current is simply proportional to the time change in E_r.

We now define the vector in unit meters as $\vec{r}_\wedge \equiv I/\langle B^2 \rangle \vec{B} - R^2 \nabla\phi$. Substituting (5) into (3) thus gives

$$\begin{aligned} \langle \vec{j} \cdot \nabla\psi \rangle = & -\frac{1}{\mu_0 v_{\mathrm{A}}^2} \langle R^2 \rangle \langle B_\theta^2 \rangle (1 + 2\hat{q}^2) \frac{\partial}{\partial t} \frac{\partial \Phi}{\partial \psi} \\ & + \sum_{s=e,i} \left[-\frac{I}{\langle B^2 \rangle} \langle B \cdot \nabla \cdot \overleftrightarrow{\Pi}_s \rangle + \langle R^2 \nabla\phi \cdot \nabla \cdot \overleftrightarrow{\pi}_s{}^{(2)} \rangle + \langle \vec{r}_\wedge \cdot \vec{R}_s^{\mathrm{nC}} \rangle \right]. \end{aligned} \quad (6)$$

It is meaningful to introduce the relative dielectric constant as follows:

$$\varepsilon_\perp \equiv 1 + \frac{\langle R^2 \rangle \langle B_\theta^2 \rangle}{\langle |\nabla\psi|^2 \rangle} \frac{c^2}{v_{\mathrm{A}}^2} (1 + 2\hat{q}^2) \equiv 1 + \kappa_{\mathrm{NC}},$$

where $\kappa_{\mathrm{NC}} \gg 1$ is called the relative neoclassical dielectric constant. This final form of the radial current is substituted into (1) to obtain

$$\begin{aligned} & \varepsilon_0 \varepsilon_\perp \langle |\nabla\psi|^2 \rangle \frac{\partial}{\partial t} \frac{\partial \Phi}{\partial \psi} \\ & = \sum_{s=e,i} \Bigg[\underbrace{-\frac{I}{\langle B^2 \rangle} \langle B \cdot \nabla \cdot \overleftrightarrow{\Pi}_s \rangle + \langle R^2 \nabla\phi \cdot \nabla \cdot \overleftrightarrow{\pi}_s{}^{(2)} \rangle + \langle \vec{r}_\wedge \cdot \vec{R}_s^{\mathrm{nC}} \rangle}_{\equiv A_s} \Bigg] + \langle \vec{j}^{\mathrm{fast}} \cdot \nabla\psi \rangle, \end{aligned} \quad (7)$$

or (1) is substituted into (6) such that Φ vanishes to obtain

$$\langle \vec{j} \cdot \nabla\psi \rangle = -\frac{\kappa_{\mathrm{NC}}}{1 + \kappa_{\mathrm{NC}}} \langle \vec{j}^{\mathrm{fast}} \cdot \nabla\psi \rangle + \frac{1}{1 + \kappa_{\mathrm{NC}}} \sum_{s=e,i} A_s \simeq -\langle \vec{j}^{\mathrm{fast}} \cdot \nabla\psi \rangle.$$

This equation clearly shows that the radial current almost completely offsets the fast-ion radial current. It is important to understand that this relationship can essentially be obtained due solely to the largeness of κ_{NC}, irrespective of a time scale in question. Note that it is the radial current that appears on the LHS, not the polarization current; however, the polarization current may be dominant relative to the components of $\langle \vec{j} \cdot \nabla\psi \rangle$ on the short time scale, as seen from (7).

In a steady state, the time-dependent LHS of (7) vanishes and the first term of the RHS also becomes nil. Replacing the non-Coulomb force term by the collisional torque term $\vec{S}_s^{\mathrm{m}}$, we have

$$-\sum_{s=e,i}\langle R^2\nabla\phi\cdot\nabla\cdot\overset{\leftrightarrow}{\pi}_s{}^{(2)}\rangle=\sum_{s=e,i}\langle R^2\nabla\phi\cdot\vec{S}_s^{\mathrm{m}}\rangle+\langle\vec{j}^{\mathrm{fast}}\cdot\nabla\psi\rangle, \tag{8}$$

where the LHS represents the viscous stress due mainly to turbulence and the RHS represents the torque input of collisions and the charge separation, respectively. This equation exhibits that externally-applied torque including the $\vec{j}\times\vec{B}$ torque and the dissipation of momentum would eventually balance out in a steady state. We should be careful that the equation indicates that the return radial current does flow and the orthogonal conduction component will predominate in $\langle\vec{j}\cdot\nabla\psi\rangle$. The existence of the radial current can be readily seen by inserting (3) without the polarization current and the non-Coulomb force directly into (8). In other words, in a steady state the radial current is composed solely of the orthogonal conduction current, and its torque is balanced by the torque created by the fast-ion current and the collisional torque.

The relationship between E_r and the radial current in the TASK/TX code

Currently, the basis equations of TASK/TX [4] essentially build on a concentric circular equilibrium, i.e. the orthogonal cylindrical coordinates (r,θ,ϕ). One of the chief characteristics of TASK/TX is that the quasi-neutrality condition is not explicitly imposed on the code; instead, the continuity equations for all charged particles are solved coupled with Gauss's law. In this sense, TASK/TX does not solve the Ampère's law given in (1) directly, but the appropriate combination of the continuity equations and Gauss's law is found to yield the Ampère's law. Reducing the basis equations in a similar manner that is shown above, we finally have

$$\varepsilon_0\left(1+\frac{c^2}{v_{\mathrm{A}}^2}\right)|\nabla\psi|\frac{\partial}{\partial t}\frac{\partial\Phi}{\partial r}=\sum_{s=e,i}\left[-\frac{I}{B^2}\langle\vec{B}\cdot\nabla\cdot\overset{\leftrightarrow}{\Pi}_s\rangle+\langle R^2\nabla\phi\cdot\nabla\cdot\overset{\leftrightarrow}{\pi}_s{}^{(2)}\rangle+RF_{s\wedge}\right]+j_r^{\mathrm{fast}}|\nabla\psi|,$$

which is almost equivalent to (7) on the LHS except for the dielectric constant. The discrepancy in the dielectric constant stems from the coordinates adopted. This result enables us to confirm that TASK/TX is essentially capable of reproducing the $\vec{j}\times\vec{B}$ torque solely within its system.

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