Estimation of the electron temperature in the plasma of a Hall thruster

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An instability known as the beam-cyclotron instability [1] leads to the growth of a longitudinal wave. This occurs when a beam is incident on a magnetized plasma in a direction orthogonal to the magnetic field. Similarly, such instability can be found in Hall thrusters [2], where the electrons drift azimuthally due to an axial electric field and a crossed radial magnetic field.

A study of the 2D dispersion relation (wave vector orthogonal to the magnetic field) has shown a mode which exists only for a discrete number of wave vectors [3]. However, recent measurements by collective light scattering have pointed out that the wave vector has a component in the magnetic field direction and that the associated dispersion relation is continuous. Accordingly, we propose here to study the dispersion relation in the 3D case.

I. The dispersion relation

The plasma at the exit of a Hall thruster may be approximated as collisionless, subject to a uniform magnetic field \( \vec{B}_0 \) parallel to the z-axis and an uniform electric field \( \vec{E}_0 \) parallel to the x-axis, in a slab geometry. The electrons are magnetized while the ions are not. In the case of a Maxwellian distribution for both ions and electrons, the kinetic theory leads to:

\[
1 + \frac{1}{k^2 \lambda_D^2} \left[ 1 + g(\omega, k, V_d, v_{the}, \omega_{ce}) \right] - \frac{1}{2k^2 \lambda_D^2} Z \left( \frac{\omega - k \cdot v_p}{\sqrt{2k \nu_{th}}} \right) = 0 \quad (1)
\]

where \( g(\omega, k, V_d, v_{the}, \omega_{ce}) \) is the Gordeev function [4]:

\[
g(\omega, k, V_d, v_{the}, \omega_{ce}) = \frac{\omega - k \cdot V_d}{k \cdot v_{the}} e^{\left( \frac{k \cdot v_{th}}{\omega_{ce}} \right)^2} \sum_{n=-\infty}^{\infty} Z \left( \frac{\omega - k \cdot V_d - m \omega_{ce}}{k \cdot v_{the}} \sqrt{2} \right) I_m \left( \frac{k \cdot v_{the}}{\omega_{ce}} \right) \quad (2)
\]

and \( Z(x) \) is the plasma dispersion function, \( Z'(x) \) its derivative and \( I_m(x) \) the modified Bessel functions of first kind. \( \omega, k \) and \( k_\perp = \sqrt{k_x^2 + k_y^2} \) are the pulsation, the wave vector and the orthogonal wave vector respectively. \( V_d = \frac{E_0}{B_0}, v_p, v_{the} \) and \( v_{thi} \) are the drift, the beam
the electron thermal and the ion thermal velocities. \( \lambda_{De} \) and \( \lambda_{Di} \) are the electron and ion Debye length.

In order to find the parameters of interest, equations (1) and (2) are normalized to \( \lambda_{De}, \omega_{pe} \) the ion plasma pulsation, and \( c_{s,0} = \sqrt{\frac{k_s T_e}{M_e}} \) the sound speed. \( \hat{M} = \frac{M_i}{m_e} \) (=2.4e4 for Xenon) and \( \hat{T} = \frac{T_i}{T_e} \) are introduced and the parameters become \( \hat{k} = k \lambda_{De}, \hat{\omega} = \frac{\omega}{\omega_{pe}}, \hat{V}_d = \frac{V_d}{c_{s,0}}, \hat{\omega}_e = \frac{\omega_e}{\omega_{pe}} \).

Thus, equations (1) and (2) transform to:

\[
1 + \frac{1}{k_z^2} \left[ 1 + g(\hat{\omega}, \hat{k}, \hat{V}_d, \hat{\omega}_e, \hat{M}) \right] = \frac{1}{2k_z^2 \hat{T}} Z \left( \frac{\hat{\omega} - \hat{k} \cdot \hat{\omega}_e}{\sqrt{2 \hat{M} \hat{T}}} \right) \tag{3}
\]

\[
g(\hat{\omega}, \hat{k}, \hat{V}_d, \hat{\omega}_e, \hat{M}) = \frac{\hat{\omega} - \hat{k} \cdot \hat{V}_d}{\hat{k} \cdot \sqrt{2 \hat{M}}} e^{-\frac{\left( \frac{\hat{k} \cdot \hat{\omega}_e}{\hat{\omega} \cdot \hat{k}} \right)^2}{2}} \sum_{m=-\infty}^{\infty} \left( \frac{\hat{\omega} - \hat{k} \cdot \hat{V}_d - m \hat{\omega}_e}{\hat{k} \cdot \sqrt{2 \hat{M}}} \right) I_m \left( \frac{\hat{k} \cdot \hat{\omega}_e}{\hat{\omega}} \right)^2 \tag{4}
\]

It is important from equations (3) and (4) to notice that \( \hat{V}_p, \hat{\omega}_p, \hat{\omega}_e \) and \( \hat{T} \) are the parameters of interest. Their consequence on \( \omega \) will be studied in the next section after solving equation (3).

II. Numerical solving of the dispersion relation

Equation (3) is a rather complicated differential equation, not easily solved. However, an approximation can be made using results from collective scattering [2]. In fact, the pulsation has found to be \( \omega = 6.10^7 \text{rad/s} \) at maximum for a corresponding wave vector of \( k_z = 12000 \text{rad/m} \). On the other hand, the drift velocity and the electron cyclotron frequency are estimated to be \( V_d = 7.10^5 \text{m/s} \) and \( \omega_{ce} = 3.10^9 \text{rad/s} \). From these results, it is clear that \( k_z V_d >> \omega \) and \( |k_z V_d + m \omega_{ce}| >> \omega \) for \( m > 0 \). Whereas for \( m < 0 \), there is some ambiguity depending on the exact value of \( k_z, V_d \) and \( \omega_{ce} \) yet the approximation is still considered. Thus, the Gordiev function (4) is now constant in \( \omega \) and equation (3) reduces to \( Z'(x) = \text{cst} \). The inverse \( Z' \) function can be found, by using a method which is not described here, allowing to calculate \( \omega \). Once calculating, \( \omega \) is reintroduced in equation (4) and recalculated in order to increase the accuracy.

Collective scattering measurements indicate a linear dependence of \( \omega \) with \( k \) leading to a group velocity equal to 3410m/s, not far from the expected sound speed \( c_{s,0} \). In order to get such a curve, a large value of \( \hat{k} \) is required and 0.03 will be taken. \( \hat{k} \) can be set as null while \( \hat{k} \) is the variable and usually lies between \( 0 \) and \( 1 \). The contribution of \( v_p \) to the pulsation is a
Doppler shift and has a small effect on the growth rate. Consequently, this quantity does not affect the group velocity and is considered null.

The dependence of the other parameters on the group velocity is studied by linearly fitting \( \dot{\omega}(\hat{k}_\gamma) \) for \( \hat{k}_\gamma \in [0.05; 0.4] \). The fitted slope called \( v_g \) is plotted versus the drift velocity \( V_{d,\text{num}} \) and for different \( \omega_{ce} \) in Fig. 1. The ratio of the temperature \( T \) was fixed to 0.06.

From Fig. 1, it is clear that \( v_g \) is quite constant and close to unity with \( V_d \) and \( \omega_{ce} \). However, as the dispersion relation is less linear when \( \omega \) increases to highest values of \( \omega_{ce}=45-55 \), \( v_g \) is seen to differ from one. Because of the linearity of the dispersion relation observed by collective scattering, values higher than 45 will be eliminated. Accordingly, the slope is found to be constant with the \( V_d \) and \( \omega_{ce} \) and to only depend on \( T \).

The dependence of \( v_g \) as a function of \( T \) is shown on Fig. 2. The ion temperature in the azimuthal direction was found \([5]\) to be low (0.06eV) compare to the electron temperature (15eV). Thus, the ratio \( T \) is believed to be small and \( v_g \) can be equal to 0.9\( \times c_s \). Consequently, the ratio of the temperature can be calculated, knowing the experimental group velocity. In the next section, a model will be developed and should presumably allow determining the electron density \( n_e \).

### III. The analytical model: evanescent magnetic field and cold ions

In the limit of a zero magnetic field, Schmitt \([6]\) has demonstrated that equation (3) should be replaced by \( \xi Z(\xi) \), where \( \xi = -\frac{V_d}{v_{th,e}} \). The mandatory conditions are that \( -\frac{kV_d}{\omega_{ce}} \) tend to infinity while \( \xi \) remains constant. These limits are not fully justified but are a starting point. In addition, a more justified limit can be taken: the cold ions limit. In that case, the right hand side of equation (3) can be re-written as \( \frac{2\hat{T}\hat{k}^2}{(\hat{\omega} - \hat{k} \hat{V}_p)^2} \). Taking these two limits into account, \( \omega \) can be obtained analytically depending on the parameters \( T_e, n_e \) and \( V_d \).
Using this model, equation (3) can be fitted with $V_d$ being the fitted parameter. $T$ and $v_p$ are taken as null. $k_x$ and $k_z$ are set as null and 0.03 respectively while $k_y$ vary from 0.1 to 1. The fitted value $V_{d,\text{model}}$ is plotted versus the input values $V_{d,\text{num}}$ on Fig. 3 for different values of $\omega_{ce}$. Except for small values of $\omega_{ce}$, $V_{d,\text{model}}$ is not far from the first bisector, indicating that the model fits quite well equation (3). Hence, by un-normalizing the model, a fit of the experimental dispersion relation would allow to determinate $T_e$ and $n_e$ knowing $V_d$.

IV. Comparison to experiment
A dispersion relation obtained by collective scattering for the azimuthal mode is presented on Fig. 4. Its fit gives a group velocity of 3410 m/s. Therefore, using the Xenon mass ($2.175 \times 10^{-25}$ kg), a value of 20eV is found for the $T_e$ using results from part II. Unfortunately, the electron density cannot be estimated with the model from part III because no curvature is visible. However, recent measurements which are not yet fully understood have shown a saturation of the frequency and would allow measuring $n_e$.

V. Conclusion
The 3D dispersion relation has been solved numerically and its initial slope is shown to be equal to $0.9 \times c_s$. A model was developed to fit equation (3) in order to find $T_e$ and $n_e$. Collective scattering measurements allow measuring group velocity and $T_e$ is estimated to be equal to 20eV neglecting $T_i$. Unfortunately, $n_e$ cannot be found because no curvature is visible. The authors would like to thank CNES and Snecma for their financial support.

References
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