On the validity of quasilinear theory

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We consider the distribution function in (x,v) space for a one-dimensional beam–plasma system initially uniform spatially ; the beam corresponds to a bump on the tail of the velocity distribution. To describe the saturation of the weak warm beam–plasma instability where resonant wave–particle interaction plays a role, we solve the Vlasov–wave equations [5]

$$\partial_t f + v\partial_x f + \varepsilon \mathrm{Re}\Big(\mathrm{i}\sum_{m=1}^{M} \beta_m \zeta_m \mathrm{e}^{\mathrm{i}(k_m x - \omega_m t)}\Big)\partial_v f = 0, \tag{1}$$

$$\dot{\zeta}_m = \mathrm{i}\varepsilon \frac{\beta_m}{k_m}\frac{1}{L}\int_0^L \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}(k_m x - \omega_m t)} f(t,x,v)\,\mathrm{d}v\mathrm{d}x, \tag{2}$$

where ζ_m is the complex envelope for wave m, with wavenumber k_m, phase velocity $v_m = \omega_m/k_m$ and pulsation $\omega_m = \omega_\mathrm{p}\sqrt{1+3k_m^2\lambda_\mathrm{D}^2}$. The coupling $\varepsilon = \sqrt{2\eta/(1+\eta)}$ is determined by the ratio $\eta = n_\mathrm{b}/n_\mathrm{p}$ of beam density n_b to plasma density n_p. Let $\Delta v_m = |v_{m+1} - v_{m-1}|/2$, $u_0 = \min(v_m)$, $u_1 = \max(v_m)$; $\Delta v_\mathrm{spec} = u_1 - u_0$ is the phase velocity width of the wave spectrum.

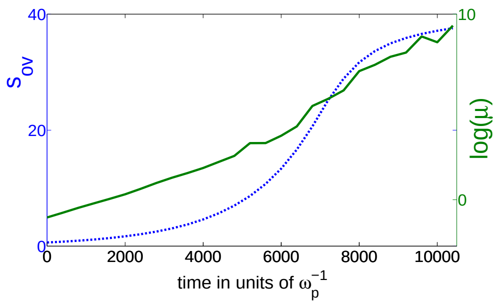
The saturation of the beam–plasma system was first predicted theoretically [6, 4] by considering the wave–particle interaction as perturbative and neglecting all mode couplings in (1)-(2), except for their effect on the space averaged distribution function $\bar{f}$. This leads to the quasilinear (QL) equations coupling $\bar{f}(t,v)$ and the waves power spectrum $\psi(t,v)$

$$\partial_t \bar{f} = \partial_v (D_\mathrm{QL}(t,v)\partial_v \bar{f}) \quad \text{and} \quad \partial_t \psi = 2\gamma_\mathrm{L}(t,v)\psi, \tag{3}$$

where $\gamma_\mathrm{L}(t,v) = \frac{\pi}{2}\frac{\eta}{1+\eta}\frac{1}{k^2}\partial_v \bar{f}(t,v)$ and $D_\mathrm{QL}(t,v) = \pi\frac{\eta}{1+\eta}\frac{1}{k^2}\psi(t,v)$ are the instantaneous Landau growth rate and QL diffusion coefficient, while $\psi(v_m)\Delta v_m = k_m|\zeta_m|^2/2$.

Note that (3) satisfies the local momentum conservation law, $\partial_t(\bar{f} - \partial_v\psi) = 0$, which follows for (1)-(2) in the dense spectrum limit $\Delta v_\varphi \to 0$ from the locality in v of wave–particle interaction [5]. This conservation law holds even if both γ_L and D_QL are rescaled by an arbitrary fuction $\alpha(t,v)$; thus its validity does not warrant the validity of (3).

For small enough initial waves amplitudes and a correspondingly small enough Δv_φ (typical Δv_m), the initial QL regime (regime IQL) is characterised by $\mu \ll 1$, $K_\mathrm{D} \ll 1$ and $\mathscr{B} \gg 1$, where



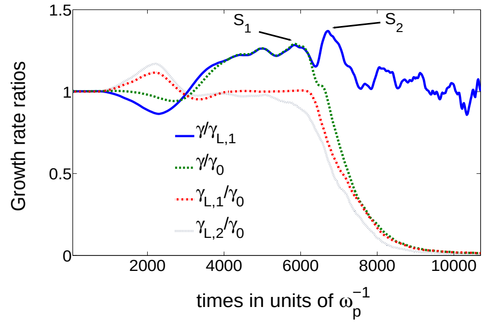


Figure 1: (left) Evolution of nonlinearity parameters μ (continued green line) and $s_{\rm ov}$ (dotted blue line). (right) Evolution of wave energy growth rate $\gamma(t) = (2E_{\rm w}(t))^{-1}{\rm d}E_{\rm w}(t)/{\rm d}t$, where $E_{\rm w}(t) = \sum_{m=10}^{M}|\zeta(t,v_m)|^2$. Average Landau growth rates are $\gamma_{{\rm L},1}(t) = (\sum_{m=10}^{M}\Delta v_m)^{-1}\sum_{m=10}^{M}\gamma_{\rm L}(t,v_m)\Delta v_m$ and $\gamma_{{\rm L},2}(t) = (\sum_{m=10}^{M}|\zeta(t,v_m)|^2)^{-1}\sum_{m=10}^{M}\gamma_{\rm L}(t,v_m)|\zeta(t,v_m)|^2$. The initial value is $\gamma_{\rm L}(0) = 10^{-3}$.

$\mu = (\gamma_{\rm L}\tau_{\rm D})^{-1}$, $K_{\rm D} = \tau_{\rm ac}/\tau_{\rm D}$ is a Kubo number, and $\mathscr{B} = \tau_{\rm D}/\tau_{\rm discr} = 8\pi^{-1/3}s_{\rm ov}^{-4/3}$ is linked to the resonance overlap parameter $s_{\rm ov} = 2\Delta v_{\rm trap}/\Delta v_{\varphi}$. Here $\Delta v_{\rm trap}$ is the typical trapping width of a wave. The Dupree time $\tau_{\rm D} = (k^2 D_{\rm QL})^{-1/3}$ defines the particle autocorrelation time, while $\tau_{\rm ac} = (k\Delta v_{\rm spec})^{-1}$ is the wave autocorrelation time, and $\tau_{\rm discr} = (k\Delta v_{\varphi})^{-1}$ is the time it takes a resonant particle to resolve the separate Doppler frequencies of the modes.

For $\mu \ll 1$, particles have a quasi-ballistic motion so that the integration of perturbations along the unperturbed characteristic curves, leading to the QL equations (3), is a valid approximation. Nevertheless there is a crossover to the strongly nonlinear regime $\mu \gg 1$ before a plateau can form in $\bar{f}$. However, though the QL near-ballistic assumption ceases to hold, the central question about the validity of QL equations per se remains open in the strongly nonlinear chaotic regime, denoted SNL and characterized by $\mu \gg 1$, $K_{\rm D} \ll 1$ and $\mathscr{B} \ll 1$. [1]

We prove theoretically that the diffusive picture, with coefficient $D_{\rm QL}$, applies in the chaotic regime SNL to the nonlinear self-consistent wave–particle interaction, by showing that, when the plateau has formed in f, the source term in (2) vanishes, so that mode coupling becomes negligible, the waves complex amplitudes are quenched, and the particle dynamics lands in a non-self-consistent stage where (as we show below) the wave spectrum meets the assumptions on which the particle velocity diffusion process rests [5] : independent phases and a non-peaked power spectrum.

Numerically, our self-consistent Vlasov simulations start from regime IQL to reach regime SNL, passing through an intermediate (nonlinear) regime INL – where $\mu \gtrsim 1$, $K_{\rm D} \ll 1$ and $\mathscr{B} \simeq 1$ – where weak renormalization effects and strong nonlinear wave coupling are expected.

In our first set of simulations, the initial f_0 is such that $\partial_v\gamma_{\rm L}(0,v) = 0$. Initial wave amplitudes are small while initial phases are drawn randomly, independently, uniformly on the circle.

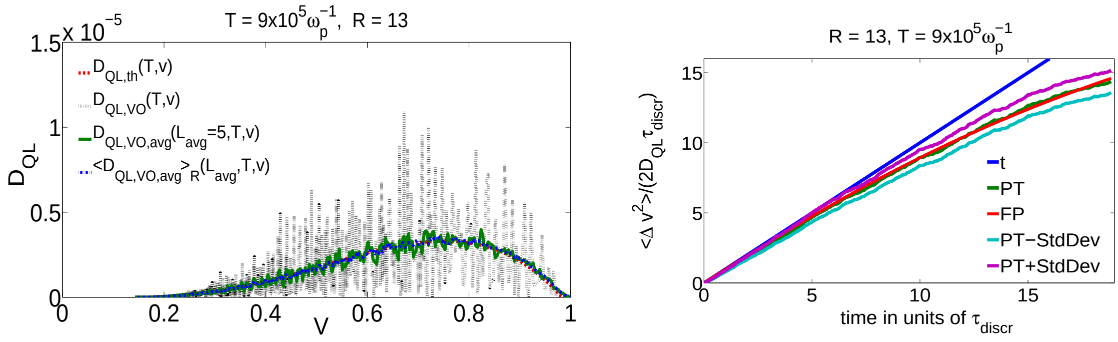


Figure 2: (left) Final diffusion coefficient for a single realization (continued grey line), windowed average over 11 nearby velocities (continued green line), ensemble average of window average (dotted blue line) and momentum conservation law prediction (dashed red line). (right) Scaled quadratic moments of particle velocity deviation. (FP): Fokker-Planck equation, (PT): 100 test particles in 13 realizations of the wave complex amplitudes $\zeta_m(T)$.

The code was validated in the linear phase by checking the numerical versus theoretical wave growth, and in the nonlinear phase by controlling conservation laws (mass, momentum, energy, L^1-norm, ...) [1]. Figure 1 shows that the theoretical linear growth of the waves coincides perfectly with the numerical one until $t = 2000\,\omega_{\rm p}^{-1} = 2/\gamma_{\rm L}(0)$, ensuring an initial linear stage where $\mu \ll 1$. For $2000\omega_{\rm p}^{-1} < t < 7000\omega_{\rm p}^{-1}$ we observe an intermediate regime where the waves growth departs from Landau's linear approximation : as noted by [3], mode coupling is present, and this intermediate stage corresponds to the transition regime INL. A striking fact is the growth rate enhancement by a factor greater than 1.2 for $4000\omega_{\rm p}^{-1} < t < 7000\omega_{\rm p}^{-1}$, reaching 1.36 at $t = 6731\omega_{\rm p}^{-1}$ ($\mu \simeq 56$, $s_{\rm ov} \simeq 20$), confirming the saturation value emerging in [3]. After it increased up to 1.36, this enhancement factor decreases, which means that the enhancement process breaks down in the INL regime when resonance overlap becomes large enough, as was also observed for the supra-QL behaviour of the diffusion coefficient in non-self-consistent dynamics [2, 5]. Nonlinear saturation is reached for $t > 7000\omega_{\rm p}^{-1}$, with f plateau set up by $t = 8400\omega_{\rm p}^{-1}$.

To characterize the SNL plateau regime, we performed a second set of simulations, with a smaller Δv_φ and final $\Delta v_{\rm D}$. In figure 2-(left) we analyse the velocity profile of the SNL diffusion coefficient. The smooth red curve is the robust prediction from integrating the local momentum conservation law $\partial_t(\bar{f} - \partial_v \psi) = 0$ with initial and boundary conditions $\partial_v \psi(0,v) = 0$ and $\psi(t,u_0) = 0$. The jagged grey line is the result for a single realization of initial data. A moving average over $2L_{\rm avg}+1$ nearby waves smoothes this profile, as shown by the green line for $L_{\rm avg} = 5 \lesssim 1/\mathscr{B}$. Averaging over a statistical ensemble ($R = 13$ realizations of the same f_0 and $|\zeta_m(0)|^2$, with random $\varphi_m(0)$) shows excellent agreement (blue line) with the (almost identical, red) conservation law prediction for the final $D_{\rm QL}$ and ψ. At the middle $v_{\rm c} \simeq 0.57$

of the waves phase velocity range, and at time $T = 9\ 10^5 \omega_{\rm p}^{-1}$ in the plateau regime, the values $\langle s_{\rm ov} \rangle_R(T, v_{\rm c}) \simeq 15$, $\mathscr{B} \simeq 0.15$, $\langle K_{\rm D}^{-1} \rangle_R(T, v_{\rm c}) \simeq 110$, and $\langle \mu \rangle_R(T, v_{\rm c}) \simeq 10^5$ confirm that the system is in regime SNL where chaotic diffusion applies.

We assess the validity of the diffusive model for particle motion in the plateau regime, by observing the spreading of $N = 100$ test particles for each of the $R = 13$ realizations at $T = 9\ 10^5 \omega_{\rm p}^{-1}$. These particle trajectories are obtained from their equations of motion with fixed ζ_m. We compare the moments of particle velocities with those of the solution to the Fokker-Planck equation (3), using the $D_{\rm QL}(T, v)$ displayed in figure 2-(left), with an initial Dirac distribution at velocity $v_{\rm c}$. Figure 2-(right) shows the good agreement of moments. The departure of both PT and FP from the straight line shows that particles are chaotically transported through the plateau and may hit the KAM boundaries of the chaotic (x, v) domain.

For QL estimates to hold, the wave spectrum must satisfy a modulus and a phase assumption. The first one requires the wave power spectrum not to have holes wider than about $\Delta v_{\rm D} = (k\tau_{\rm D})^{-1} = \Delta v_\varphi / \mathscr{B}$. Figure 2-(left) shows that, even if nearby waves have intensities strongly inhomogeneous with respect to velocity, their averages over a range $L_{\rm avg} \Delta v_\varphi \lesssim \Delta v_{\rm D}$ (green line) make both ψ and $D_{\rm QL}$ appear smooth enough and close to the conservation law prediction. Therefore the square modulus requirement, absence of holes wider than $\Delta v_{\rm D}$, is met. The second requirement is randomness of wave phases. In figure 3, the statistical distribution of difference between final and initial phase appears coherent, which means that $\varphi_m(0)$ and $\varphi_m(T)$ are strongly correlated. As our random initial phases are independent, we may expect near-independence for final ones. In a way, the transition to chaos turns out to be orderly.

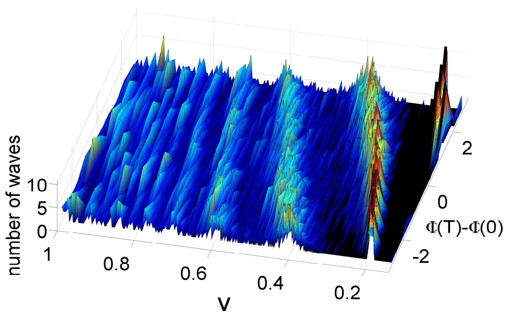


Figure 3: Distribution of difference between final and initial phase of the waves.

The fulfillment of both conditions on $\zeta_m(T)$ explains the agreement in figure 2-(right).

References

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