Complete plasma and sheath solution for Tonks-Langmuir models with warm ion sources

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The problem of Tonks-Langmuir-type collisionless discharges [Phys.Rev. **34**(6), 876 (1929)] is a very old and basic one but, unfortunately, has hitherto been solved only with various simplifying assumptions facilitating solution for restricted ranges of validity and particular applications. The mathematical and computational difficulties of the problem at hand are so severe [K.-U. Riemann, in 62nd Ann. Gaseous Electronics Conf., (2009)] that during last 80 years it was only treated for the limiting cases $\varepsilon \to 0$ (which represents the ratio of Debye length over discharge characteristic length) and/or the normalized ion source temperature $\tau \to 0$, but never solved properly for arbitrary finite values of both quantities. In the present paper we present a reliable method of solution not subject to any restriction of this kind, and demonstrate the results to be applicable in a practically unlimited range of plasma parameters appropriate for experimental and fusion-relevant plasmas.

Introduction

A general mathematical formulation of the problem can be expressed as a task to find the unknown quantity $\Psi(\Phi)$, which physicaly represents the inverse of the electric field in the plasma and sheath regions (with Φ the local electrostatic potential). Our strict mathematical formulation of the problem can be condensed into the general integro-differential equation

$$\varepsilon^2 n(\Phi)\frac{1}{\Psi^3}\frac{d\Psi}{d\Phi} = 1 - \lambda\int_0^{\Phi}\Psi(\Phi')\mathscr{K}\left(\tau(\Phi'-\Phi)\right)d\Phi' \tag{1}$$

with $\mathscr{K}\left(\tau(\Phi'-\Phi)\right)$ a prescribed singular kernel (which depends of the particular ion source velocity distribution taking role in a given physical scenario and is based on trajectory integration of the Boltzmann kinetic equation), $n(\Phi)$ a prescribed function (which physically represents the electron density distribution), ε and τ properly chosen plasma parameters and λ the eigenvalue of the problem, which must be determined on the basis of proper boundary conditions.

Model and basic equations

The general formulation of the problem for a plane-parallel symmetric discharge consists in simultaneously solving Boltzmann's kinetic equation for the ion velocity distribution function

(VDF) $f_i(x,v)$,

$$v\frac{\partial f_i}{\partial x} - \frac{e}{m_i}\frac{d\Phi}{dx}\frac{\partial f_i}{\partial v} = S_i(x,v) \ , \tag{2}$$

and Poisson's equation,

$$-\frac{d^2\Phi}{dx^2} = \frac{e}{\varepsilon_0}(n_i - n_e) \ . \tag{3}$$

The source term $S_i(x,v)$ on the right-hand side of Eq. (2) describes microscopic processes assumed for a particular scenario of interest, x is the Cartesian space coordinate, v is the particle velocity, e is the positive elementary charge, m_i is the ion mass, $\Phi(x)$ the electrostatic potential at position x, ε_0 is the "vacuum dielectric constant" and n_i and n_e are the ion and electron densities, respectively.

The geometry of the symmetric problem considered is schematically shown in Fig. 1. The plates at $x \pm L$ are assumed to be perfectly absorbing and electrically floating. The electrostatic potential $\Phi(x)$ is assumed to be monotonically decreasing (for $x > 0$) and is defined to be zero at $x = 0$. The ion source term $S_i(x,v)$, describing the ion generation due to electron-neutral impact, is assumed to be of the form

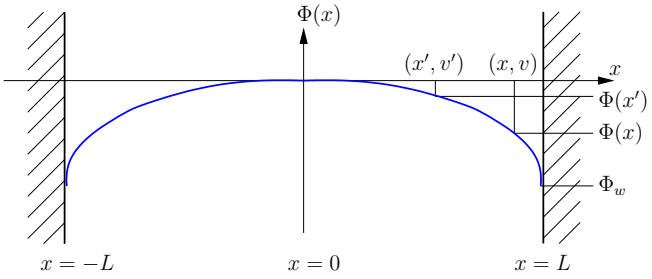

Figure 1: The geometry and coordinate system.

$$S_i(v,x) = R n_n n_e(x) f_n\left(\frac{v}{v_{T_i}}\right) \ , \tag{4}$$

where R is the (constant) ionization rate, n_n is the (uniform) number density of neutrals and the $f_n(v/v_{T_i})$ is the neutral-particle VDF (which is quite general at this point). The electrons follow the Boltzmann distribution, i.e., the electron number density $n_e(x) = n_0 \exp\left(\frac{e\Phi(x)}{kT_e}\right)$, with n_0 the electron density at $x = 0$. In the present work we focus on the case of Maxwellian ion source velocity distribution, i.e., f_n is a Maxwellian. We introduce the dimensionless quantities

$$u = \frac{v}{\sqrt{2}c_s} \ , \ \frac{e\Phi(x)}{kT_e} \to \Phi(x) \ , \ \frac{x}{L} \to x \ , \ n = \frac{n_i}{n_0} \ , \ j = \frac{J_i}{n_0 c_s} \ , \ \tau = \frac{T_e}{T_i} = \frac{1}{T_n} \ , \ c_s = \sqrt{\frac{kT_e}{m_i}} \ , \tag{5}$$

in which Poisson's equation (3) becomes

$$\begin{aligned} &B\int_0^1 dx' \exp\left[\Phi(x') - \Phi(x)\right] \exp\left[\frac{\tau}{2}\{\Phi(x') - \Phi(x)\}\right] \mathrm{K}_0\left\{\frac{\tau}{2}|\Phi(x') - \Phi(x)|\right\} \\ &= 1 - \varepsilon^2 \exp(-\Phi)\frac{d^2\Phi}{dx^2} \end{aligned} \ , \tag{6}$$

where $\varepsilon = \lambda_D/L$ is an arbitrary parameter and $\lambda_D = \sqrt{\varepsilon_0 k T_e/e^2 n_0}$ is the electron Debye length at the centre. Equation (6) describes the potential profile for arbitrary source temperature. For the floating potential of the wall we obtain the relation

$$\exp(\Phi_w) = 2\pi\sqrt{\frac{m_e}{m_i}}\sqrt{\frac{T_i}{T_e}}B\int_0^1 dx' \exp[\Phi(x')] \; . \tag{7}$$

Results

It is a relatively simple task to formulate the problem but a rather demanding one to perform numerical calculations. This is due to many reasons ranging, e.g., from numerical reliability and stability to the CPU cost per simulation case. It thus seems to be a logical path to be pursued here is to continue on our previous works [3, 4] and to extend previously well established results to the extent possible via employing Maxwellian ion-source velocity distribution of arbitrary T_n and without any limit of the system length, i.e., without any restriction of ε.

The following results were obtained with the unified program code that still maintains both ($\varepsilon = 0$ and $\varepsilon > 0$) models. The computational grid and almost the entire code remain the same, although the numerical models are different. Our results for the potential profiles with $L = 1$ are shown in Fig. 2. High resolution of the grid when approaching $x = 1$ enabled calculations for small $\varepsilon(\leq 0.0006)$.

For calculating the second derivative we implemented piecewise Lagrangian polynomial interpolation [5] of order 2 or 3 in subintervals with small Φ gradients; 5 point

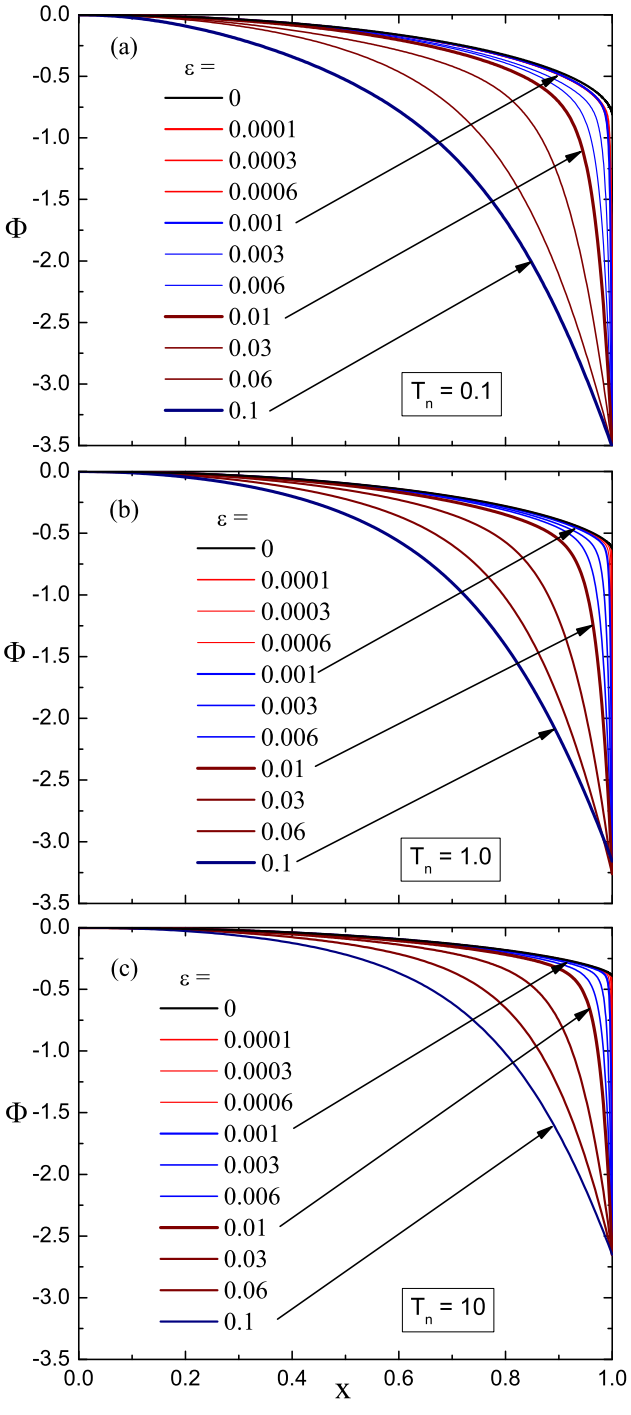

Figure 2: Potential profiles for various ε and T_n.

Lagrangian interpolation has been used as a basis for derivative. For the last point 4-point second derivative of inverse Lagrangian interpolation is used. Although this type of approximation is often considered to be too expensive for numerical computation, it possesses beautiful symmetry and in a modified (weighted) form is comparable in speed to other approximations.

The main difference as compared to the $\varepsilon = 0$ case is that now the wall potential Φ_w is a free parameter that can be arbitrary. For gases like hydrogen, used in fusion, Φ_w can be approximated by the floating wall condition Eq. (7) for $\varepsilon = 0$ case. With the boundary condition $\Phi(1) = \Phi_w$ the integro-differential equation (6) becomes a relaxation problem in a numerical sense although the whole system is still floating. For initialization we have implemented the following function that assures a monotonous initial "shot" to the endpoint: $\Phi[i] = \Phi_w \left[1 - \exp\left(i/N\right)\right] / \left(1 - \exp(1)\right)$, disregarding λ_1 and λ_2. When initial profile has relaxed and converged we employ additional dual iteration of the potential profile and endpoint Φ_w with a similar smooth stepping technique to assure correct bounary condition (7) for a given gas.

Conclusion

Our investigation covers a wide range of ion source temperatures and a wide range of ε. In fact, this is the first investigation of this kind using the analytic-numerical method. Another method assume is PIC simulation (see e.g., [6]). The problem of the intermediate region between sheath and plasma as provided by Riemann's rules [7] for the finite ion source temperature turns out to be rather unavailing at least in fusion plasmas.

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