Modification on the gyrokinetic quasi-neutrality condition due to large flow

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Introduction

Reduced kinetic models such as gyrokinetic models are constructed by phase space transformation from particle phase space to guiding-centre phase space. The quasi-neutrality condition or the Poisson equation for electrostatic potential as well as the Vlasov equation should be modified in the reduced models[1]. Since guiding-centre density which is obtained from integration of guiding-centre distribution function is generally different from particle density due to finite Larmor radius effects, it is needed to represent the particle density in terms of guiding-centre things. It is called push-forward representation of particle density[2]. Recently a reduced phase space Lagrangian with large $\mathbf{E}\times\mathbf{B}$ flow which is observed in transport barrier regions such as a tokamak edge in an H-mode regime and an internal transport barrier in a reversed shear tokamak was derived by modifying the standard guiding-centre transformation[3]. The phase space Lagrangian for dynamics of a single particle with mass m and electric charge q is given by

$$L_p = q\mathbf{A}^*\cdot\dot{\mathbf{X}} + \frac{m}{q}\mu\dot{\xi} - H, \tag{1}$$

where $\mathbf{Z} = (\mathbf{X}, U, \mu, \xi)$ are the guiding-centre coordinates, $\mathbf{A}^* = \mathbf{A} + (m/q)U\hat{b}$, H the guiding-centre Hamiltonian. The model can be regarded as a natural extension of the standard model to the large flow regime because the symplectic part of the Lagrangian is the same as the standard guiding-centre one formally, while the Hamiltonian H is not the standard one. In conventional models with large flow, $\mathbf{A}^*$ includes a flow term which changes the standard symplectic structure when the flow is time-varying[4]. The phase space Lagrangian similar to the present one is also found in [5]. We derive the push-forward representation of particle density associated with the modified guiding-centre transformation. The representation can be derived by two ways. One is to expand the exact representation perturbatively. The other one is a variational method by which the representation is derived from the single particle Lagrangian (1). In this paper we assume that a magnetic field is independent of time and consider only the electrostatic case.

Perturbative expansion of the exact representation

A general particle fluid moment is defined by

$$m_{kl}(\mathbf{r}) \equiv \int \left(\frac{mw^2}{2B}\right)^k v_\parallel^l f \delta^3(\mathbf{x}-\mathbf{r}) d^3\mathbf{x} d^3\mathbf{v}, \tag{2}$$

where f is the particle distribution function, $w = |\mathbf{v}_\perp - \mathbf{v}_E|$ the perpendicular particle velocity in a frame moving with the $\mathbf{E}\times\mathbf{B}$ velocity $\mathbf{v}_E$, $v_\parallel$ the parallel particle velocity. The particle fluid moment can be written in terms of the guiding-centre distribution function F and the push-forward transformation associated with the guiding-centre transformation $\mathsf{T}_{\mathrm{GC}}^{-1*}$ as

$$m_{kl}(\mathbf{r}) = \int d^6\mathbf{Z} \mathscr{J}(\mathbf{Z}) \left[\mathsf{T}_{\mathrm{GC}}^{-1*}\left\{\left(\frac{mw^2}{2B}\right)^k v_\parallel^l\right\}\right](\mathbf{Z}) F(\mathbf{Z}) \delta^3(\mathsf{T}_{\mathrm{GC}}^{-1}\mathbf{x}-\mathbf{r}), \tag{3}$$

where $\mathscr{J} = B_\parallel^*/m$ is the Jacobian with $B_\parallel^* \equiv \hat{b}\cdot\mathbf{B}^*$ and $\mathbf{B}^* \equiv \nabla\times\mathbf{A}^*$, and $\mathsf{T}_{\mathrm{GC}}^{-1}\mathbf{x} = \mathbf{X}+\rho+\rho_E+\cdots$ denotes the particle position in the guiding-centre phase space with $\rho = \hat{b}\times\mathbf{w}/\Omega$, $\rho_E = \hat{b}\times\mathbf{v}_E/\Omega$ and $\Omega = qB/m$. The velocity variables are related with the guiding-centre variables as

$$\frac{mw^2}{2B} = \mu - G_1^\mu + \cdots, \quad v_\parallel = U - G_1^U + \cdots, \tag{4}$$

where G_1^μ and G_1^U are μ and U components of the vector field generating the guiding-centre transformation at first order in ε ($\varepsilon \sim \rho/L$), respectively. Equation (3) is the formal exact representation. Assuming that the $\mathbf{E}\times\mathbf{B}$ velocity is subsonic $v_E \sim \varepsilon^{1/2} v_{ti}$ (v_{ti} the ion thermal speed) and expanding the above exact representation perturbatively, we have the push-forward representation of m_{kl} up to $O(\varepsilon^2)$[6],

$$m_{kl} = M_{kl} + \frac{1}{2}\nabla\cdot\left[\frac{\nabla_\perp M_{k+1l}}{q\Omega}\right] + (k+1)\nabla\cdot\left[\frac{M_{kl}}{B\Omega}\nabla_\perp\varphi\right] - k\mathbf{v}_E\cdot\frac{\hat{b}\times\nabla M_{kl}}{\Omega}, \tag{5}$$

where M_{kl} is a guiding-centre fluid moment defined by

$$M_{kl} \equiv \int \mu^k U^l F \mathscr{J} dU d\mu d\xi. \tag{6}$$

The last term on the right hand side of Eq. (5) does not appear in the one obtained from the standard gyrokinetic theory in which $v_\perp$ is used for the magnetic moment[7]. The exact representation usually used in the standard gyrokinetic theory is given by[8]

$$m_{kl}(\mathbf{r}) = \int d^6\bar{\mathbf{Z}} \mathscr{J}(\bar{\mathbf{Z}}) \left[\mathsf{T}_{\mathrm{GC}}^{-1*}\left\{\left(\frac{mv_\perp^2}{2B}\right)^k v_\parallel^l\right\}\right](\bar{\mathbf{Z}}) [\mathsf{T}_{\mathrm{Gy}}^* \bar{F}](\bar{\mathbf{Z}}) \delta^3(\mathsf{T}_{\mathrm{GC}}^{-1}\bar{\mathbf{X}}-\mathbf{r}), \tag{7}$$

where $\bar{\mathbf{Z}}$ denotes the gyro-centre coordinates and $\mathsf{T}_{\mathrm{Gy}}^*$ the pull-back transformation associated with the transformation from the guiding-centre phase space to the gyro-centre phase space.

Although it seems to be different from Eq. (3), direct correspondence between these two is found by considering the alternative exact representation for the standard gyrokinetics[9]. For $k=l=0$ we have the push-forward representation of particle density,

$$n = N + \frac{1}{2}\nabla\cdot\left[\frac{\nabla_\perp P_\perp}{q\Omega B}\right] + \nabla\cdot\left[\frac{N}{B\Omega}\nabla_\perp\varphi\right], \tag{8}$$

where $n \equiv m_{00}$, $N \equiv M_{00}$ and $P_\perp \equiv BM_{10}$ are particle density, guiding-center density and guiding-centre pependicular pressure, respectively. This representation is the same as the standard one formally.

Variational derivation

The push-forward representation of particle density can be derived from the single particle Lagrangian (1). To this end, we consider a functional derivative of the action functional $I = \int_{t_1}^{t_2} L dt$ with a Lagrangian for the Vlasov-Poisson system[10],

$$L = \sum\int d^6\mathbf{Z}'\mathscr{J}(\mathbf{Z}')F(\mathbf{Z}',t')L_p[\mathbf{Z}_p(\mathbf{Z}',t';t),\dot{\mathbf{Z}}_p(\mathbf{Z}',t';t),t] - \int d^3\mathbf{x}\frac{1}{4\mu_0}\mathsf{F}:\mathsf{F}, \tag{9}$$

where $\sum$ denotes a sum over species, $\mathbf{Z}_p(\mathbf{Z}',t';t)$ denotes the guiding-centre coordinates of the particle at t with the initial condition $\mathbf{Z}_p(\mathbf{Z}',t';t') = \mathbf{Z}'$, μ_0 is permeability of vacuum and the electromagnetic field tensor F is defined by $\mathsf{F}_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ with the covariant four vector potential $A_\mu = (-\varphi/c, \mathbf{A})$ and the four gradient operator $\partial_\mu = ((1/c)\partial_t, \nabla)$. $\delta I/\delta\varphi(\mathbf{r}) = 0$ yields a reduced Poisson equation in which charge density is expressed in terms of the guiding-centre things. We can obtain the push-forward representation of particle density by comparing the charge density in the reduced Poisson equation with the particle charge density qn. In our guiding-centre model the general push-forward representation of particle density is given by

$$n(\mathbf{r}) = \frac{1}{q}\int d^6\mathbf{Z}\mathscr{J}(\mathbf{Z})F(\mathbf{Z})\frac{\delta H(\mathbf{Z})}{\delta\varphi(\mathbf{r})}, \tag{10}$$

where H is the guiding-centre Hamiltonian and $\delta H(\mathbf{Z})/\delta\varphi(\mathbf{r})$ is the functional derivative of H with respect to $\varphi(\mathbf{r})$. First we consider

$$H = \underline{q\varphi} + \frac{m}{2}U^2 + \mu B \underline{-\frac{m}{2}v_E^2} + \underline{\frac{m}{2q}\mu\hat{b}\cdot\nabla\times\mathbf{v}_E} \tag{11}$$

which is valid in well localised transport barrier regions with subsonic flow[3]. It is similar to the standard gyrokinetic Hamiltonian in the long wavelength limit. Since it is not H itself but $\delta H/\delta\varphi$ that is necessary for the push-forward representation of particle density, it is sufficient to keep in mind underlined terms which include φ. Substituting the above Hamiltonian into Eq. (10) and integrating by parts yield Eq. (8). The first underlined term leads to the first term in

Eq. (8), the second leads to the term including φ and finally the third leads to the term with $P_\perp$. Thus the variational method is more transparent and useful than the perturbative expansion of the exact representation if the guiding-centre Hamiltonian is known. When the symplectic part of L_p contains φ as in the conventional formulations with large flow[4], we have to consider its contribution in addition to the guiding-centre Hamiltonian.

Transonic flow case

In this section we derive the push-forward representation in the transonic flow case by the variational method. When the $\mathbf{E}\times\mathbf{B}$ velocity is comparable to the thermal velocity, we have to use the following Hamilatonian[3],

$$H = q\varphi + \frac{m}{2}U^2 + \mu B - \frac{m}{2}v_E^2 + \frac{m}{2q}\left(\mu + \frac{mv_E^2}{2B}\right)\hat{b}\cdot\nabla\times\mathbf{v}_E, \tag{12}$$

which is still limited to the localised transport barrier case. From Eq. (10) we obtain

$$n = N + \frac{1}{2}\nabla\cdot\left[\frac{1}{q\Omega B}\nabla_\perp\left(P_\perp + \frac{Nmv_E^2}{2}\right)\right] + \nabla\cdot\left[\left(1 - \frac{\hat{b}\cdot\nabla\times\mathbf{v}_E}{2\Omega}\right)\frac{N}{B\Omega}\nabla_\perp\varphi\right]. \tag{13}$$

The additional terms appear as corrections to the polarisation density. The first one is the correction to $P_\perp$. The second one is the correction by the vorticity which gives a term proportional to enstrophy density. They are nonlinear to φ because they come from the cubic term of v_E in the Hamiltonian.

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